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PLANE WAVES OF LIGHT--I

BY

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A BRIEF EXPOSITION
OF THE BEHAVIOR OF PLANE LIGHT WAVES GIVING
SPECIAL ATTENTION TO CERTAIN PECULIAR TYPES
WHICH ARISE FROM METALLIC REFLECTION

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PLANE WAVES OF LIGHT

I. ELECTROMAGNETIC BEHAVIOR

BY THORNTON C. FRY

1. *Scope of the paper.* It would seem that the subject of plane light waves had been exhausted. If information is wanted, the obvious thing is to go to Drude or Wood for it. So when I wanted to make some apparently routine calculations on the absorption of light by thin films of metals, I merely chose the formula that seemed suited for the purpose. The results, however, were obviously wrong; though what was wrong with them proved to be far from obvious. In fact, before I had found my mistake and corrected it, I had been led to work through the entire subject from the beginning in my own way.

In many respects the results at which I arrived in this way were different from anything contained in the usual reference books. A search of the literature has shown that few, if any, are new; but it has also confirmed the belief that they are decidedly not common knowledge. In fact, in so far as metals are concerned, the only similar work appears to have been done by E. Ketteler, in a book called *Theoretische Optik*, published in 1885. This book is not in the New York Public Library, and I have been unable to locate it elsewhere. I only know of it through Ketteler's complaint,¹ that its results had been consistently overlooked by later writers. He did not use Maxwell's Theory but he seems nevertheless to have obtained much valuable information. Among other things he claims to have discovered the properties of the "principal azimuth," which affords one basis of determining optical constants, and with which he is not usually credited.²

This being the state of the literature on the subject, a purely expository paper such as the present is probably worth while. The general outline of the paper is as follows: Starting with Maxwell's equations as data, and assuming a form of solution which ultimately turns out to be the appropriate one for plane waves, but which contains a large number of constants, the relations are presented which these constants must satisfy *because of purely mathematical requirements*. In order that this part of

¹ In a paper in *Wied. Ann.*, 67, p. 879-893; 1899.

² Other anticipatory papers are: W. Voigt, *Wied. Ann.*, 67, p. 185-200, 1899; E. E. Hall, *Phys. Rev.*, 15, pp. 73-106, 1902; Schaefer and Gross, *Ann. d. Phys.*, 32, 1910. These deal only with dielectrics. The search, however, has not been at all exhaustive and a more comprehensive treatment may have been overlooked.

the paper may not read too much like a text book on algebra, the derivation of the relations is omitted.

Thereafter, the constants are taken up one at a time, and their physical significance discussed, after which the physical meaning of their mathematical irrelations is studied.

2. *The fundamental equations.* We begin with Maxwell's equations

$$\begin{aligned}\text{curl } E &= -\frac{\mu}{c} \frac{\partial H}{\partial t}, \\ \text{curl } H &= \frac{1}{c} \left(\sigma E + \epsilon \frac{\partial E}{\partial t} \right), \\ \text{div } E &= 0, \\ \text{div } H &= 0,\end{aligned}\tag{1}$$

and ask for the most general solution which is an exponential function of x, y, z and t . Since E and H are vectors, a solution must be defined by six equations, one for each of the components $E_x, E_y, E_z, H_x, H_y, H_z$. We choose one of them, to begin with, and call it ϕ . Then if the solution is to be of the type chosen, ϕ must be of the form

$$\phi = \Phi e^{iq(ax+by+cz-ipt)}$$

and the others must be of similar form.

The constants Φ, iqa, iqb, iqc , and ip are, so far as we know at the start, perfectly arbitrary in all six equations. However, if we substitute them in (1), we find at once that the last four must remain unchanged throughout the set, otherwise (1) would either not be satisfied *everywhere*, or else not at all times. The six equations can therefore differ only in the particular values assigned to Φ . We denote these by, $El, Em, En, Hl', Hm', Hn'$, respectively, and write the solution in the form

$$\begin{aligned}E_x &= El e^{iq(ax+by+cz-ipt)}, & H_x &= Hl' e^{iq(ax+by+cz-ipt)}, \\ E_y &= Em e^{iq(ax+by+cz-ipt)}, & H_y &= Hm' e^{iq(ax+by+cz-ipt)}, \\ E_z &= En e^{iq(ax+by+cz-ipt)}, & H_z &= Hn' e^{iq(ax+by+cz-ipt)}.\end{aligned}\tag{2}$$

No generality is lost by assuming, as we shall do hereafter, that the triplets $a, b, c; l, m, n; l', m', n'$ satisfy the laws

$$\begin{aligned}a^2 + b^2 + c^2 &= 1, \\ l^2 + m^2 + n^2 &= 1, \\ l'^2 + m'^2 + n'^2 &= 1;\end{aligned}\tag{3}$$

for q , E , and H are still arbitrary, and this allows all combinations that would otherwise be possible.

But not only must the exponents be alike; there are other relations which must be satisfied as well. They appear at once when (2) are substituted in (1), for then the variables all cancel out and leave only a system of simple algebraic equations, which, by routine algebra, can be thrown into the form³

$$\begin{aligned} al + bm + cn &= 0, \\ al' + bm' + cn' &= 0, \\ ll' + mm' + nn' &= 0; \end{aligned} \tag{4}$$

$$\begin{aligned} l' &= bn - cm, \\ m' &= cl - an, \\ n' &= am - bl; \end{aligned} \tag{5}$$

$$\begin{aligned} l &= cm' - bn', \\ m &= an' - cl', \\ n &= bl' - am'; \end{aligned} \tag{6}$$

$$\begin{aligned} a &= mn' - nm', \\ b &= nl' - ln', \\ c &= lm' - ml'; \end{aligned} \tag{7}$$

$$q^2 = hg; \tag{8}$$

$$H = \frac{iq}{h} E. \tag{9}$$

In the last two g and h stand for the combinations

$$\begin{aligned} g &= \frac{\sigma - \epsilon i p}{c}, \\ h &= \frac{\mu i p}{c}, \end{aligned} \tag{10}$$

which recur with such regularity in studying reflection that it is desirable to have special symbols for them.

³ As a convention regarding the coordinate axes, we measure positive values of z in that direction in which a righthand screw would move if rotated from x toward y .

This list contains all the relations which we shall want to use, which happens to be more than are logically required to insure that (2) is a solution of Maxwell's equations. In fact, any three of the sets (3) to (7) could be omitted without harm. But on the other hand, we have placed no unnecessary restrictions upon the solution by including the others, for they are merely the minimum requirements masquerading in a different mathematical dress.

Even when reduced to a minimum, however, there are *eight* conditions imposed upon the *thirteen* constants. Hence only *five* can be chosen at will, and it is found that these five must be chosen as follows: two (but not p, q) from among p, q, E, H ; the remaining three from among the triplets $a, b, c; l, m, n; l', m', n'$ (but not all from one triplet).

3. *Physical restrictions on the solution* (2): So far we have paid no attention to the physics of our problem, but only to its mathematics. Now, however, we are ready to begin to interpret our equations.

At the very beginning we are confronted with the fact that physical quantities are always real. Any set of values we may wish to give to our arbitrary constants—whether real or complex—is allowable *mathematically*, since the fundamental equations (1) will be satisfied; but unless these constants result in real E 's and H 's they mean nothing *physically*. Actually, very few such sets can be found, and they are trivial.

However, Maxwell's equations are linear, and we know that when such equations are satisfied by a complex solution, they must be satisfied by the real and imaginary parts of that solution separately, or by either part multiplied by any constant we please. Thus, if p, q, E, H , and all three triplets in (2) are real, E_x reduces to the sum of two terms,

$$El \cos [pt - q(ax + by + cz)]$$

and

$$-iEl \sin [pt - q(ax + by + cz)],$$

because $e^{-i\theta} \equiv \cos \theta - i \sin \theta$. The first of these is real as it stands, and the second can be made real by multiplying by i . The same is true of E_y, E_z , and the three H 's. Either set of results—that is, the one containing the cosines or the one containing the sines—is a *real physical* solution.

This process, which is well known when the constants are real, is not, however, limited to that case. It applies equally well when they are complex. It does not even require the solution to be exponential in

form. Hence, instead of a complex solution not defining *any* physical disturbance, it defines *two* such disturbances simultaneously.

4. *What the constants mean when they are real:* When all the constants in (2) are real, an observer who stationed himself at the origin of coordinates would see the disturbances as

$$E_x = El e^{-ipt}, \quad E_y = Em e^{-ipt}, \quad E_z = En e^{-ipt}$$

with a similar set for the H 's. Each of these equations, however,⁴ represents a simple harmonic oscillation of frequency $p/2\pi$. One oscillation is directed along each of the three axes; and as they all reach their maximum extension at the time $t=0$ —that is, as they are all in phase—the resultant of the three is also a simple harmonic motion along the diagonal of the parallelepiped which their maximum extensions define. It, too, is in phase with all its components.

Obviously El , Em and En are the “amplitudes” of the components; while by virtue of (3) the amplitude of their resultant is E . A glance

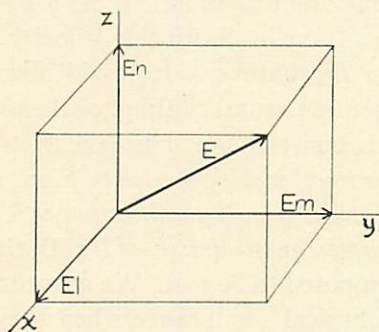


FIG. 1.

at Fig. 1 shows at once that l , m and n are the cosines of the angle which the vector E makes with the three axes: they are called the “direction cosines” of E .

Similarly, the group of H 's represents a simple harmonic oscillation of amplitude H , along a line defined by the direction cosines l' , m' , n' .

Next suppose the observer changes his position to a point x' , y' , z' . He now observes the oscillation in the form

$$E_x = El e^{i(\eta - pt)}, \quad E_y = Em e^{i(\eta - pt)}, \quad E_z = En e^{i(\eta - pt)},$$

where η stands for $q(ax' + by' + cz')$. The only difference between this

⁴ When only the real parts are considered. Unless otherwise definitely specified, what we say about the physical significance of our equations will always refer to the real part only.

result and the preceding one is, that the harmonic components reach their maximum extension at the time η/p . *The disturbance at x', y', z' is just the same as at the origin, but lags by the fraction $\eta/2\pi$ of a period.*

It follows, of course, that the same is true of the disturbances at every other point the coordinates of which satisfy the relation $q(ax+by+cz)=\eta$. This, however, is the equation of a plane. It is proved in books on analytic geometry that a perpendicular let fall upon it from the origin has the direction cosines a, b, c , and is η/q units in length. In other words: *The wave is a plane wave, since the points of like phase lie in a plane (the wave front). It is also plane-polarized since the electric and magnetic vectors never change direction.*⁵

Finally, let the observer leave the origin at the time $t=0$, and travel along the normal to the wave front with a velocity p/q . At the time η/p he will be at a distance η/q from the origin. But we have seen that it is at this time that the E -vector reaches maximum extension at this place. The phase of the disturbance at η/q when he arrives is therefore just what it was at the origin when he left. As η/q is really any distance at all, it follows that, travelling with this velocity, the observer would note no variation in E whatever. It would always be at maximum extension. It is therefore natural to think of the plane of zero phase as travelling along with him. Hence it is said that *planes of like phase travel normally to one another with a velocity p/q ; or in other words, the phase velocity of the wave is p/q , in the direction a, b, c .*

5. *Meaning of complex quantities: E or H :* In the preceding sections all constants were supposed to be real. We now turn our attention to a discussion of their physical significance when they are complex. To simplify matters we shall consider them one by one without thought for the interrelations which they bear to one another. Later on, this defect will be corrected.

If E (or H) is complex, it is best to write it in the polar form $E_0 e^{i\eta}$. Then at the origin E_x takes the form $E_x = E_0 e^{i(\eta-pt)}$, and reaches its maximum extension at time η/p . That is, a wave of "complex amplitude" $E_0 e^{i\eta}$ lags behind a wave of real amplitude E by $\eta/2\pi$ of a cycle. The amplitude is E_0 . *There is no other difference.*

Now, turning to (8), (9) and (10) we find that q is real, and therefore H/E is real, whenever $\sigma=0$; but if $\sigma>0$, H/E must be complex. Thinking of E as real, it follows that H is also real in dielectrics, but complex in metals. Hence *in dielectrics the electric and magnetic vectors are in phase, in metals not.*

⁵ Of course they change sense twice each cycle.

It can be shown that the H lags behind the E by an amount ranging from zero in dielectrics to an eighth of a cycle in perfect conductors.

6. *Meaning of complex quantities: p and q .* Complex p means a transient disturbance; that is, one which dies out as time goes on. We have no further interest in it.

If q is complex, let us write it $\kappa + i\kappa$. Then the exponential terms of (2) take the form

$$e^{i\kappa(ax+by+cz)-ipt} \cdot e^{-\kappa(ax+by+cz)}. \quad (13)$$

If the second term of this product were not present, (2) would represent a plane-polarized plane wave of light traveling in the direction a, b, c with a velocity p/κ . As it is, if this second term is thought of in connection with E , it becomes obvious that it merely modifies the amplitude in such a way that the maximum extension of E (and the same is true of H) is not as great at some points as others.

At all points on the plane $ax+by+cz=d/\kappa$, the amplitude is Ee^{-d} , as compared with E at the origin. This plane has the same direction cosines a, b, c as the wave front; hence *amplitudes as well as phases are alike at all points of a wave front when a, b, c are real.*⁶ If $d/\kappa=1$, the origin is one unit distance, in the shortest direction, from the plane under consideration. That is, the plane on which the amplitudes are E and that on which they are $Ee^{-d}=Ee^{-\kappa}$ are unit distance apart. Therefore, *in unit distance the amplitude falls off by a factor $e^{-\kappa}$: κ will be called the "damping per unit distance."*⁷

Of course, since H is proportional to E , it follows that H is damped out at the same rate, and in the same direction, as E .

⁶ The reason for this proviso will appear in sec. 8.

⁷ Damping is usually measured in optics by what is called an "extinction coefficient." There are two definitions. In our language, the usual one (Drude, Wood) is equivalent to $1/2\pi$ times the damping per *metal*-wave-length. To be strictly accurate we should add "of plane polarized light," for a reason which will appear in sec. 8. For the type of wave which we are considering at present the wave-length is $\Lambda=2\pi/K$. Hence the extinction coefficient K so defined is:

$$K = \kappa/\kappa.$$

The other (Abraham-Föppl) is equivalent to $1/2\pi$ times the damping per *ether*-wave-length. The latter wave-length is $\Lambda_0=2\pi c/p$. Hence the extinction coefficient K_0 is

$$K_0 = c\kappa/\kappa.$$

The latter definition is decidedly the better. Not only is the color of light usually described by its ether-wave-length, but, in addition, when a wave of ether-wave-length Λ_0 impinges on a metal, the velocity of propagation and therefore *the wave-length in the metal varies with the angle of incidence*. It seems much better to tie our definitions up to something that can be expected to stay fixed throughout our experiments. This also will appear in sec. 8.

Tacitly, of course, the usual definition refers to the metallic disturbance produced by normally incident light.

7. *Meaning of complex quantities: l, m, n and l', m', n' :* When l, m, n are complex they may be written

$$l = \lambda + i\lambda',$$

$$m = \mu + i\mu',$$

$$n = \nu + i\nu'.$$

Then E_x can be split up into the sum of two terms

$$E_x = E\lambda e^{iq(ax+by+cz)-ipt},$$

$$\mathbf{E}_x = iE\lambda' e^{iq(ax+by+cz)-ipt};$$

and a similar process may be carried out for E_y and E_z . Considering real parts, the terms E_x , E_y and E_z are just like (2) and represent component oscillations all of which are in phase with one another. There is only this difference: Where we knew, in connection with Fig. 1, that $l^2 + m^2 + n^2 = 1$, we do not know this about $\lambda^2 + \mu^2 + \nu^2$. Suppose, however, that we call the latter sum u^2 and define a new set of numbers L, M, N as $\lambda/u, \mu/u, \nu/u$. Then $L^2 + M^2 + N^2 = 1$ and the set E_x, E_y, E_z take the form

$$E_x = (Eu)L e^{iq(ax+by+cz)-ipt},$$

$$E_y = (Eu)M e^{iq(ax+by+cz)-ipt},$$

$$E_z = (Eu)N e^{iq(ax+by+cz)-ipt}.$$

This defines a plane wave of plane-polarized light travelling in the direction a, b, c with its E -vector constantly pointing in the direction L, M, N , its maximum extension being Eu .

As for the set $\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z$, they may be treated in exactly the same way; the result being a wave travelling in the direction a, b, c , with its E -vector pointing in the direction $\mathbf{L} = \lambda'/u, \mathbf{M} = \mu'/u, \mathbf{N} = \nu'/u$, its amplitude being Eu . ($u^2 = \lambda^2 + \mu^2 + \nu^2$). But there is this difference: the i in the equations for $\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z$ is equivalent to $e^{i\pi/2}$, and causes this vector to reach its maximum extension a quarter period later than the other. Hence when the two are superposed to give back the original oscillation they result in a rotating vector.

When l, m, n are complex the E -vector oscillates as in elliptically polarized light, the two components being of magnitude Eu and Eu and having

Attention need scarcely be called to the fact that, since the index of refraction is by definition the ratio of the velocity of light in space to that in the medium, we have the law $N = ck/p$. Taking this in conjunction with the definitions given above, we have the relations

$$q = \frac{p}{c}(N + iK_0) = \frac{p}{c}N(1 + iK).$$

the direction cosines L, M, N and $\mathbf{L}, \mathbf{M}, \mathbf{N}$, respectively. The latter lags 90° behind the former.

Let us now note that when $l = \lambda + i\lambda, \dots$, are substituted in (3) the result

$$u^2 - \mathbf{u}^2 + 2i(\lambda\lambda + \mu\mathbf{u} + \nu\mathbf{v}) - 1 = 0$$

is obtained. But as the real and imaginary parts must separately vanish, this requires that

$$\begin{aligned} u^2 &= \mathbf{u}^2 + 1, \\ \lambda\lambda + \mu\mathbf{u} + \nu\mathbf{v} &= 0. \end{aligned} \tag{11}$$

The first says that $u > \mathbf{u}$, which tells which is the larger component in the elliptical oscillation. To interpret the latter we make use of a theorem from differential geometry: that the cosine of the angle between two intersecting lines is

$$\cos \phi = \mathbf{L}\mathbf{L} + \mathbf{M}\mathbf{M} + \mathbf{N}\mathbf{N},$$

the L, M, N and $\mathbf{L}, \mathbf{M}, \mathbf{N}$ being their direction cosines. But by dividing (11) by uu it is easily seen that the angle between the directions of the two oscillating components is 90° . Hence Eu and $E\mathbf{u}$ are at right angles; $Eu > E\mathbf{u}$.

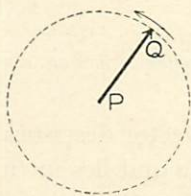


FIG. 2.

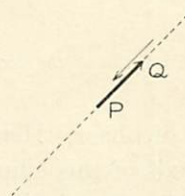


FIG. 3.

Before proceeding further, we need some scheme for representing graphically the life history of our variable forces. If P is a point of observation past which a wave of light is travelling, an arrow may be drawn from it to represent the magnitude and direction of the electric vector at a given instant. As time passes this vector changes—either in length or direction, or both—and its end-point therefore describes some sort of curve. If this curve is known, much of the life history of the vector is known; particularly if its position at the time $t=0$ and the direction in which the curve is described are given.

Thus Fig. 2 represents a vector the length of which never changes. At the time $t=0$ it was in the direction PQ , (as shown by the heavy arrow) and thereafter rotated counter-clockwise about P . The electric force in a circularly-polarized beam of light would be of this form.

Fig. 3 represents a vector which always lies along the line PQ , but which changes magnitude (and sense). At the time $t=0$ it was of length PQ (half its maximum extension) and was receding (as shown by the arrow). The electric vector of plane-polarized light behaves in this way.

Now let us use this sort of diagram to illustrate what happens when l, m, n are complex. We know that the resultant oscillation is the sum of two oscillating components, (Fig. 4a) at right angles to one another

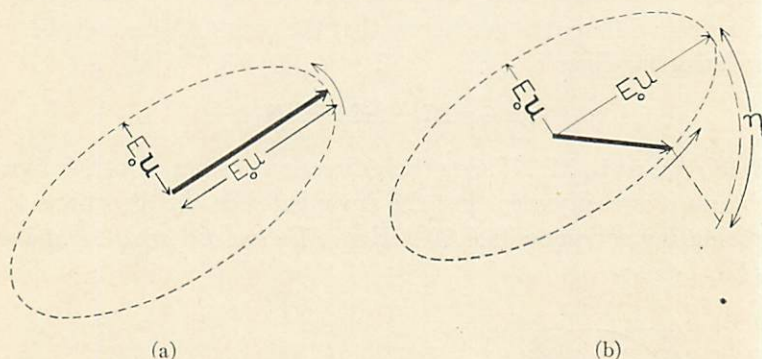


FIG. 4. The behavior of the electric vector when its direction cosines are complex, (a) in a dielectric; (b) in a metal.

and 90° out of phase. Hence the resultant vector describes an ellipse. The major axis of this ellipse is of length $2Eu$ and lies in the direction L, M, N ; while the minor axis is of length $2Eu$ and lies along L, M, N . The direction of rotation is from L, M, N toward L, M, N (or, as we may say, from the positive major toward the positive minor axis). It is shown by the arrow in Fig. 4a.

If E is a complex quantity $E_0 e^{i\eta}$, the entire argument regarding l, m, n goes through exactly as before. The only effect of the η is to change the phase of the elliptical oscillation, as shown in Fig. 4b.⁸

Of course, all these statements apply equally well to l', m', n' and the accompanying H .

⁸ The exact position of the arrow is determined as follows: a circular arc is struck with the major axis as radius, and the angle η laid off as shown. Then a perpendicular is dropped to the major axis. This cuts the ellipse in the required point. If η were negative the angle would be laid off in the other sense (that is, toward $E_0 u$ instead of away from it).

8. *The meaning of complex quantities: a, b, c :* If we write

$$a = \alpha + i\alpha,$$

$$b = \beta + i\beta,$$

$$c = \gamma + i\gamma,$$

and for the moment assume q to be real, the exponential terms in (2) take the form

$$e^{i q (\alpha x + \beta y + \gamma z) - i p t} \quad e^{-q (\alpha x + \beta y + \gamma z)}. \quad (14)$$

Obviously the first of these factors defines a progressive wave motion, and the second one "damps it out." A complex q led to two similar factors and defined a damped wave even when a, b, c were real. But there is a vital difference between the two cases: for when a, b, c were real both the planes of like phase and the planes of like amplitude were defined by the same equation $ax + by + cz = \text{constant}$; whereas here the planes of like phase are defined by $\alpha x + \beta y + \gamma z = \text{constant}$, and the planes of like amplitude by a different equation $\alpha\alpha x + \beta\beta y + \gamma\gamma z = \text{constant}$. It is therefore not necessary that they be parallel. In fact, if we denote $\alpha^2 + \beta^2 + \gamma^2$ by w^2 , and $\alpha\alpha + \beta\beta + \gamma\gamma$ by w^2 it is easily seen from (3) that $w^2 = 1 + w^2$, and that $\alpha\alpha + \beta\beta + \gamma\gamma = 0$. As in the case of complex l, m, n , the second of these shows the two lines to be at right angles. *The planes of like amplitude are normal to the wave-front.*

Our equation (14) is still not in a form strictly analogous to (13), however. For in (13) the a, b, c are direction cosines; in (14) the corresponding triplets α, β, γ and $\alpha\alpha, \beta\beta, \gamma\gamma$ must first be divided by w and w to make direction cosines of them. Suppose we call the results A, B, C and A, B, C . Then (14) becomes

$$e^{i q w (Ax + By + Cz) - i p t} \quad e^{-q w (Ax + By + Cz)},$$

in which qw obviously plays the same role as the κ of (13), and qw as the κ . It follows at once that *the normal to the wave-front is in the direction A, B, C , and the wave travels along this normal with a velocity p/qw . As $w > 1$, this velocity is less than the velocity p/q with which a wave travels when a, b, c are real. Moreover, since the planes of like phase are at right angles to the planes of like amplitude, the wave suffers no damping in its direction of propagation; but the amplitude is damped out at the rate of qw per unit length in the direction A, B, C , which lies in the wave front.*

In other words, an observer who stationed himself in a wave-front and moved *with* it would observe no change of amplitude; but if he began moving about *in* it, he would find that the amplitude was not everywhere constant.

Such waves actually exist physically in the case of "total reflection." Take for example the case of Fig. 5, in which the upper medium may be thought of as glass and the lower as air.

As the velocity of ordinary light in glass is less than that in air, the distance between the wave fronts a and b of a beam incident in the glass is less than that between the wave-fronts of its refracted beam in air.

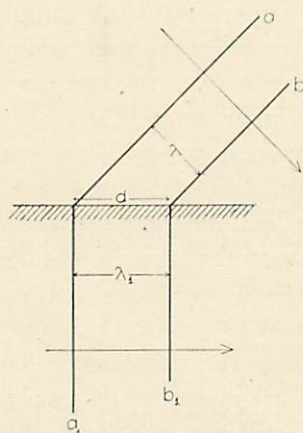


FIG 5.

Now if the electromagnetic boundary conditions at the boundary are to be satisfied these wave-fronts must match up, which they can only do provided the lower set have a greater tilt than the upper. By no manner of manipulation of the lower set, however, can the distance d be made less than λ_1 , while the upper set can be tilted still further, in which case d would be still smaller. There is therefore a limiting angle of incidence (the angle of total internal reflection) beyond which no *ordinary* light waves in air can match up with the waves in the glass. Hence when light waves impinge on the glass-air surface at still greater angles, the disturbance in the air must move with a velocity less than "the velocity of light."

If *all* the energy is reflected, there can be none flowing into the air. Hence if the phase velocity implies a stream of energy, as we are accustomed to think it does, this energy stream can have no component

normal to the boundary. Therefore the velocity vector is parallel to the boundary. Obviously, however, the intensity of the disturbance in the air must be the same at all points of a plane parallel to the boundary. Hence the damping takes place perpendicular to the boundary; which makes it perpendicular to the velocity.

So far in this discussion q has been assumed to be real, so that what has been said applies to dielectrics only. The same argument in the main can be carried out if q is complex. In that case, when the exponent $iq(ax+by+cz)-ipt$ in (2) is separated into its real and imaginary parts, it is found that the disturbance is again a wave travelling in one direction and damped out in another. However the planes of like phase now have the equation

$$(\alpha\kappa - \alpha\kappa)x + (\beta\kappa - \beta\kappa)y + (\gamma\kappa - \gamma\kappa)z = \text{const.},$$

while the planes of like intensity have the equation

$$(\alpha\kappa + \alpha\kappa)x + (\beta\kappa + \beta\kappa)y + (\gamma\kappa + \gamma\kappa)z = \text{const.}$$

By a certain amount of rather tedious algebra it can be shown that the phase-velocity is $p/(w^2\kappa^2 + w^2\kappa^2)^{1/2}$, which is smaller than the velocity p/κ which a wave has when a, b, c are real; and that the damping is $(w^2\kappa^2 + w^2\kappa^2)^{1/2}$ per unit length, which is greater than in the more usual case; and that the directions of the two are not at right angles. Hence an observer who fixed himself upon a wave-front and travelled with it would observe a decreasing amplitude as he progressed; and he would also find the amplitude to vary, if, having "frozen" the wave he moved about from point to point of the wave-front.

This sort of wave is set up in a metal whenever a plane wave of light shines upon its surface.

No matter what the angle of incidence may be, the entire surface is uniformly illuminated. If the metal were in the form of a very thin sheet, it is physically obvious that the *under* surface would be uniformly illuminated too. In other words, the planes of like intensity would be parallel to the surface of the metal. The same is true whether the sheet be thin or not. As energy is dissipated in the metal, there must be a component of energy-flow normal to its boundary. Hence, if the energy flows in the direction of the phase velocity, concerning which sec. 11 will leave us somewhat in doubt, though not in a sense which invalidates our present argument, the velocity vector must be inclined to the surface. The wave-front therefore cannot be at right angles to the planes of like amplitude.

Intuitively, therefore, we could have inferred that such waves exist; but we could hardly have concluded that the wave length of the light within the metal varies with the angle of incidence, nor that the extinction coefficient does. But it is now obvious that both these things are true. For when the light is *normally* incident, the refracted beam obviously has its planes of like phase coincident with its planes of like intensity. It therefore has real values of a, b, c . At all other angles of incidence the sets of planes are inclined, which implies complex a, b, c . We have seen that complex a, b, c *always implies decreased velocity* (wave length) and *increased damping* (extinction).⁹

The study of reflection, however, though it gave rise to the present paper, is not included in its scope except to the extent of illustrating that "waves of complex a, b, c " are physical realities. There are, however, other general properties of these "hybrid waves," as we shall call them hereafter, which we have not yet derived.

9. *The orthogonality relations:* So far we have made no use of the relations (4) to (7), and indeed, when the quantities are all real, there is little to say about them. By comparing (4) with (12) it becomes obvious that the three vectors, direction of propagation, electric intensity and magnetic intensity, are mutually at right angles; and (5), (6), and (7) merely repeat this fact in another form. But when the quantities are complex the equations tell us a great deal more.

To start with, we notice that if any one of the three triplets a, b, c ; l, m, n ; l', m', n' is complex, one other at least must be. For suppose a, b, c to be real and l, m, n to be complex. Then by (5)

$$\begin{aligned} l' &= (bv - c\mu) + i(bv - c\mu) = \lambda' + i\lambda', \\ m' &= (c\lambda - av) + i(c\lambda - av) = \mu' + i\mu', \\ n' &= (a\mu - b\lambda) + i(a\mu - b\lambda) = \nu' + i\nu'. \end{aligned} \tag{30}$$

⁹ To this argument the objection may be raised that in absorbing media like dye-stuffs the damping is determined solely by the distance the beam travels, and not by the angle at which it enters. It can be shown mathematically that this result is *approximately* true of weak absorbers, the approximation being so nearly exact that it is doubtful whether a variation could be detected physically. But in strong absorbers like metals an oblique beam may fall off in intensity much more rapidly *per unit travel* than one that is normally incident.

In the air-disturbance outside a totally reflecting glass surface the condition is similar, in a way. At the *exact* angle of total reflection the intensity of the "refracted" beam is the same (theoretically, at least) at a point a millimeter from the surface as it is at the surface; but as the angle is exceeded by larger and larger amounts the intensity at this point rapidly decreases. Of course, the two things (metallic and total reflection) are not strictly analogous, for there is no propagation into the air and there is into the metal.

Unless λ', μ', ν' are all zero the triplet l', m', n' is complex; and λ', μ', ν' can only vanish simultaneously if all three of the quantities λ, μ, ν are zero, in which case l, m, n would not be complex as supposed.

There are, then, three possible situations: (a) all three triplets real; (b) one real and two complex; (c) all three complex. Of these, the first has been adequately considered already, and the last can be shown to result only from the superposition of two waves of the second type, just as elliptically polarized light results from the superposition of two beams of plane polarized light. Hence it is only necessary for us to consider the second case, where one triplet is real. The problem again resolves itself into three cases according as the real triplet is a, b, c ; l, m, n ; or l', m', n' .

The mathematical relations which we need are the same whichever triplet is real, for the equations (4) to (7) are all perfectly symmetrical; but it happens to be simplest physically to think of a real a, b, c , which corresponds to the perfectly familiar case of elliptically polarized light. We begin, then, with this case. From (4):

$$(a\lambda + b\mu + c\nu) + i(a\lambda' + b\mu' + c\nu') = 0.$$

Since the real and imaginary parts of this equation must vanish separately it is really equivalent to *two* equations, and by dividing these equations by u and u' respectively we obtain

$$\begin{aligned} aL + bM + cN &= 0, \\ aL' + bM' + cN' &= 0. \end{aligned}$$

Hence the velocity vector is at right angles to *both* components of the electric force: that is, it is at right angles to the plane in which the electric force lies. A similar result follows from the second of equations (5). Hence the velocity vector is at right angles to the plane of H also. This, however, can only happen provided *in elliptically polarized light the electric and magnetic vectors rotate in one and the same plane*.

Next we inquire as to the shape and situation of the H -ellipse. It is very easy to see from (30) that

$$\lambda\lambda' + \mu\mu' + \nu\nu' = 0 :$$

in fact when the real parts of the middle members of (30) are multiplied by λ, μ and ν respectively, the coefficients of a, b , and c all vanish at once. This equation (divided, of course, by the proper quantities u and u' in order to produce direction cosines) says that the directions L, M, N and L', M', N' are at right angles. Hence *in elliptically*

polarized light the major axis of the H -ellipse coincides with the minor axis of the E -ellipse. Of course, the other pair must also coincide.¹⁰

Finally, by routine algebra which we need not go through it can be shown that $u' = u$ and $u' = u$. That is, in elliptically polarized light the H - and E -ellipses are similar.

The situation is that shown in Fig. 6, the part (a) of which corresponds to a dielectric, with E and H in phase, and the part (b) to a metal where H lags behind. The heavy arrows represent instantaneous positions of the E - and H -vectors at a given point of space. As time

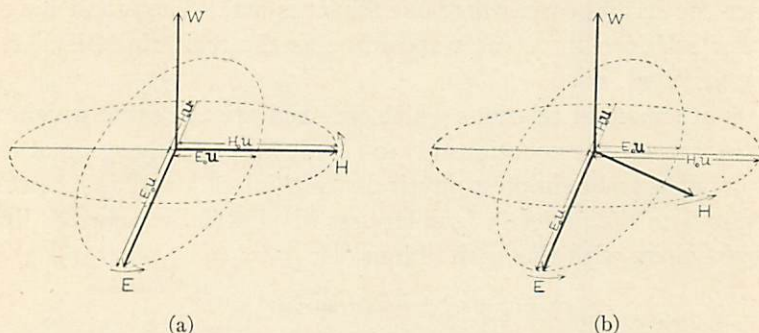


FIG. 6. Illustrating the relation of the electric and magnetic vectors of elliptically polarized light to the velocity and damping, (a) in a dielectric; (b) in a metal.

passes they rotate in the horizontal plane, and in the direction indicated, with a frequency of revolution equal to the frequency of the light. At some other point off in the direction of propagation w the same situation exists, except that E and H do not reach the marked positions quite so soon. In other words, the "phase" to which the heavy arrows correspond arrives later and later at points further and further above the plane of the diagram. In addition, in the metallic case (b), there is a shrinking of everything as we travel off in the direction w , due to the absorption of energy.

There are still two things about Fig. 6 which have not been shown to be true: first, that the E and H vectors rotate in the same direction; second, that w points in the direction in which a right handed screw would move if turned from E toward H . To prove these things requires only algebraic juggling of a sort with which we can dispense.

Finally it should be said that, though E and H have been shown rotating counter-clockwise this may be reversed without affecting the

¹⁰ The E - and H -ellipses of Fig. 6 have been drawn equal as well as similar, but the dimension lines indicate their true magnitudes. As these magnitudes vary with the units of measure used, they are of no particular importance.

direction of propagation w , *provided both are reversed*. In fact, exactly this change results if all three triplets are replaced by their conjugate-complexes—a process which does not affect the *real* components which define w and the *major* axes, but changes the signs of the imaginary terms upon which the *minor* axes depend.

10. *Hybrid waves*: The argument of section 9 is based upon the assumption that a, b, c is the real triplet. But the only equations used in the argument were (3), (4), (5), (6) and (7); except that in defining phase relations in metals (9) was also used. Now if a, b, c is replaced by l, m, n ; l, m, n by l', m', n' ; and $l' m' n'$ by a, b, c , the equations (3) to (7) reappear in exactly the same form. The same is true if a, b, c ; l, m, n ; l', m', n' are replaced by l', m', n' ; a, b, c ; l, m, n . Hence we must conclude that what we have learned in section 9 would have been equally true *in a mathematical sense* if we had dealt with either of the other cases; though of course its *physical interpretation* might be vastly different. There is only one place where caution would have to be observed: when phase relations in metals were under discussion equation (9) was used, and it does not take part in this cyclic permutation of triplets.

Now all this being true, it is certainly much simpler to re-phrase the results of sec. 9 in general terms and re-interpret them, than it would be to carry the same discussion through twice more. We therefore adopt this plan, and agree to use the following expressions for still further simplification: “first triplet” for the real one and “second” and “third” for the others in the proper cyclic order:

First	$a, b, c,$	$l, m, n,$	$l', m', n',$
Second	$l, m, n,$	$l', m', n',$	$a, b, c,$
Third	$l', m', n';$	$a, b, c;$	$l, m, n;$

“major components” of a triplet for the real terms which give rise to the major axis of “its ellipse”; and “minor components” for the others. Then our results phrase themselves as follows:

- a. The first triplet defines *one* vector of unit length.
- b. The second defines two vectors—one by its major components; the other by its minor components. The same is true of the third triplet.
- c. The major and minor components of the same triplet are at right angles. The minor may be either to the right or left of the major; but whichever is true for the second triplet is true for the third also.
- d. The two major components are equal in length, and so are the two minors.

e. The ellipses of the second and third triplets have their axes interchanged.

f. They both lie in a plane normal to the vector defined by the first triplet.

g. If the major component of the second triplet were rotated toward the major component of the third, it would cause a right-handed screw to move in the direction of the first.

When the first triplet is l, m, n , the second l', m', n' , and the third a, b, c , the physical interpretation of these statements leads to Fig. 7(a). The E -vector has *real* direction cosines: it therefore oscillates as in Fig. 5 (plane polarized light). In Fig. 7(a) it is shown pointing toward

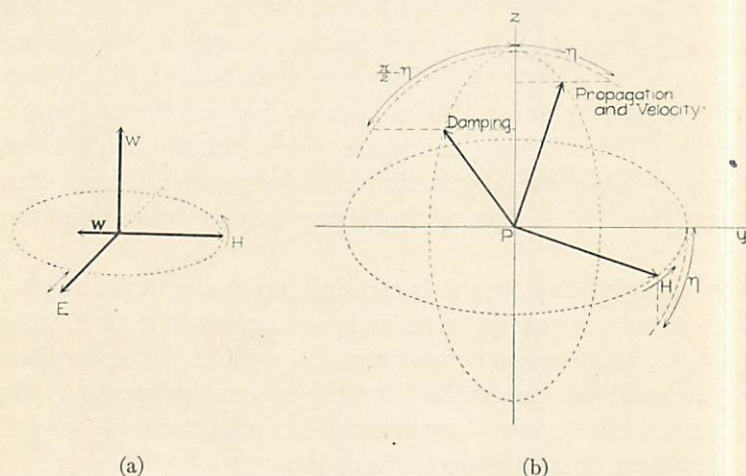


FIG. 7. Illustrating the relation of the electric and magnetic vectors of hybrid light to the velocity and damping, (a) in a dielectric; (b) in a metal. Magnetic vector rotating.

the reader. The H -vector (the second triplet) rotates as in elliptically polarized light. It is shown rotating in the vertical plane, with its instantaneous position pointing toward the right. The minor axis is vertically upward. To the third triplet also corresponds a pair of vectors (b),¹¹ equal in length to the two axes of the H -ellipse (d); situated with respect to one another in the same relative fashion as the axes of H (c); at right angles to E (f); and so placed that if the major H -axis were rotated toward w it would cause a right handed screw to move in the direction of E (g). These vectors derived from a, b, c have been quite thoroughly discussed in sec. 8. The major one w points in the direction of the phase velocity, and its length defines the magnitude of that

¹¹ These letters refer to the list of similarly marked statements above.

velocity by means of the formula p/qw . The minor one w points in the direction in which the amplitude decreases, and its length measures the damping per unit length by means of the formula qw .

At points in the direction w the situation is just the same, except for delayed phase. At points in the direction w it is just the same, except that everything is shrunk to a smaller scale; and conversely at points in the opposite direction the diagram must be magnified. The wave front is the plane Ew . The planes of like amplitude are parallel to Ew . All this will be found in section 8.

If the direction of rotation of H be reversed, which causes the minor axis to point downward instead of upward, w must point toward the right.

This is the sort of disturbance set up outside a prism when plane polarized light with its E -vector normal to the plane of incidence is totally reflected by it.

Now turning to the case of a metallic medium, for which q is complex, we obtain the diagram of Fig. 7(b). The disposition of the axes of the ellipses is the same as before. So is the E -vector, which is omitted to avoid confusion. But the presence of a *complex* amplitude factor $H = H_0 e^{i\eta}$ causes the phase to be retarded by an amount η , as explained in secs. 5 and 7. PH , not Py , is the instantaneous position of the vector.

So far everything is as before. But when we come to the *third* triplet, a, b, c , we are confronted by the necessity of taking account of the complex q which also contains the factor $e^{i\eta}$. Its effect is to "retard" the velocity vector by a "phase" η around its "ellipse"; and the damping vector also. That is, the term $e^{i\eta}$ treats the velocity vector (the normal to the wave front) and the damping vector (the normal to planes of like amplitude) *which do not rotate* just as it treats the instantaneous position of a rotating vector. I cannot justify presenting the algebraic proof of this here; but there should be no need of it anyway, since the symmetry of the three cases is a sufficient proof.

I must point out, however, that the velocity and damping are no longer at right angles. As such a wave can be produced by letting plane-polarized light fall on a metallic surface this is to be expected, for as was noted in sec. 8 the planes of like amplitude cannot be at right angles to the wave front.

This type of light is most peculiar. The E -vector behaves as in plane polarized light; the H -vector as in elliptically polarized light, except that its plane of rotation is not the wave front, but is at right angles to the wave front. In a sense, at least, it is not a transverse vibration.

It is for this reason that I call it "hybrid."

The third case where l', m', n' are real leads to Fig. 8(a) when q is real. About this figure nothing need be said, except to point out that interchanging the positions of E and H has reversed both w and w . But when q is complex different set of conditions arise, due to the fact that the rotating vector E now *leads* the oscillating vector H^{12} : that is, E/H contains the term $e^{-i\eta}$, since H/E has the term $e^{+i\eta}$. Hence PE , Fig. 8b, is shifted *toward* its minor axis, not away from it. But a, b, c is still multiplied by q in (2) just as before, and q still has the factor

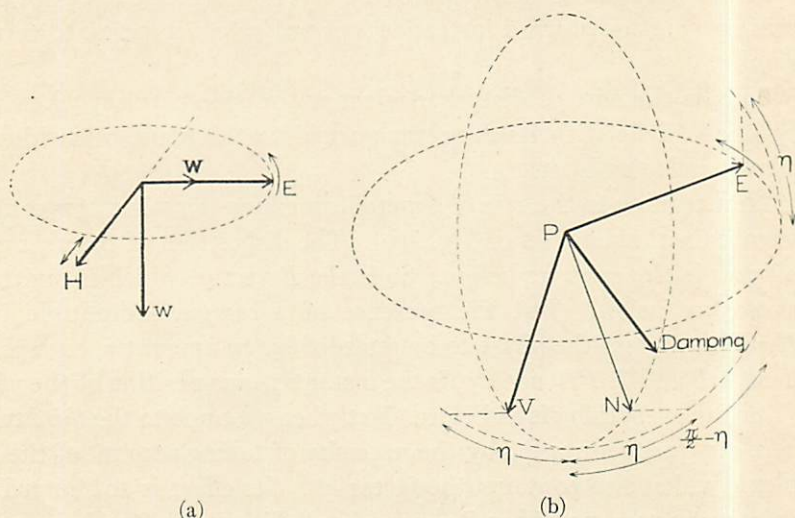


FIG. 8. Illustrating the relation of the electric and magnetic vectors of hybrid light to the velocity and damping, (a) in a dielectric; (b) in a metal. Electric vector rotating.

*e*ⁱⁿ. Hence the velocity and damping factors are "retarded," as before. This unfortunately spoils what would otherwise be a beautiful general law: for in every other case the direction of propagation is at right angles to the position of the rotating vector at that instant when the oscillating vector is at maximum extension. In the present case that direction (*PN*) is not the direction of propagation, but the two are mirror images in the major axis of the *a*, *b*, *c* ellipse.

This type of light, for q real, results from total reflection at a dielectric when the incident light is plane-polarized with H normal to the plane of incidence. The complex q corresponds to reflection of similar light from a metal.

¹² All our vectors are shown at the instant when the oscillating vector (if there is one) is at its maximum extension. The analogies are more striking when this is done.

11. *The propagation of energy.* Whether Poynting's vector defines an actual physical flow of energy is a matter of some doubt. A uniform magnetic field crossed by a uniform electric field gives rise to an invariable Poynting vector at right angles to both, and therefore to a constant stream of energy, if that is what the Poynting vector means, though why such a stream should exist is not clear. As a matter of fact, the basis for so interpreting the Poynting vector is rather flimsy. It is merely that the rate of accumulation of energy inside any prescribed volume is just what it would be *if* the energy flowed in that way. The relation is not unique, however; for to any Poynting vector could be added the velocity vector of any motion which an incompressible fluid might undergo, and the argument would be just as strong for the resultant as for the original Poynting vector.

There is, however, nothing better in electromagnetic theory, wherefore we must either talk of *it* or not mention the energy stream at all. That energy flows cannot be doubted, and the desire to learn something about it is too natural to allow us to neglect it altogether.

The exact Poynting formula is

$$S = \frac{c}{4\pi} [EH], \quad (50)$$

the brackets indicating a vector product. E and H are here instantaneous values of the forces—not their amplitudes—and S is the instantaneous rate of flow. It may vary with time in either magnitude, direction, or both. Naturally all the quantities are real.

Suppose, now, that E and H are at right angles, and that both are simple harmonic functions of time, and *in phase*. That is, suppose $E = E_0 \cos pt$ and $H = H_0 \cos pt$. Then S is $(c/4\pi) E_0 H_0 \cos^2 pt$. It is zero whenever pt is an odd multiple of $\pi/2$ (twice in each cycle of E and H) and reaches a positive maximum in between; but it never changes sign or direction. As the average value of $\cos^2 pt$ is $\frac{1}{2}$, it follows that the *average* energy stream is $S = (c/8\pi) E_0 H_0$. This, then, is the rate at which energy is transported *in the long run*.

Now plane-polarized light in a dielectric is of this sort, and there is nothing unnatural about supposing that energy actually flows past a fixed observer in this way; that is, in the direction in which the wave is traveling (phase velocity), though not in a steady stream, but in two spurts during each period of the light.

Next, suppose we had an E and an H , at right angles to one another and simple harmonic as before, but this time 90° out of phase.¹³ Then

S would be $(c/4\pi)E_0H_0 \cos pt \sin pt$. This S changes sign four times each cycle; during the first quarter cycle energy flows out, during the second back, and then repeats. This, too, is quite reasonable: for if E changes sign and H does not the direction of flow ought (we are quite satisfied to believe) also to change sign, and this or the reverse condition exists at end of each quarter cycle. Obviously the average transport is zero.

Finally, returning to (50), suppose the vector E is the resultant of any two vectors E_1 and E_2 , and H the same. Then obviously

$$\begin{aligned} \frac{4\pi}{c}S &= [(E_1+E_2)(H_1+H_2)] \\ &= [E_1H_1] + [E_1H_2] + [E_2H_1] + [E_2H_2]. \end{aligned} \quad (51)$$

That is, S is the resultant of four vectors, one formed from each combination of component E 's and H 's. This is a purely mathematical result; it has nothing whatever to do with the question of whether these combinations could separately exist as physical waves. I mean this: It is obvious that if we begin with a physically possible plane-polarized wave and split its E into two components, say $E_1 = \frac{1}{3}E$ and $E_2 = \frac{2}{3}E$, without splitting the H , the two combinations E_1 and H , and E_2 and H , have no physical existence. Nevertheless, if an energy flow is computed for each hypothetical pair, the sum will be the true energy of the original wave. This is all that (51) implies.

Take, for example, the case of a plane-polarized wave in a metal, for which $E = E_0 \cos pt$ and $H = H_0 \cos (pt - \eta)$. By expanding the $\cos (pt - \eta)$ we find that H is the sum of two vectors. The combination of E with H_1 gives a pulsating flow just like that for similar light in a dielectric, the average transport being $S_1 = (c/8\pi)E_0H_0 \cos \eta$; while the combination of E with H_2 gives a surge and resurge in the same direction, with no resultant transport. In the long run, therefore, energy is transported at the rate S_1 . Now let us return to Fig. 5(b), which represents just this case. The figure is drawn for the instant when E is at maximum extension, and the length of the H -vector at this instant is $H_0 \cos \eta$, which is exactly the amplitude of the component H_1 . In other words: *In plane-polarized light the long time average of transport of energy, both as to amount and direction, is the vector product of the E -vector at its maximum extension, by the value of the H -vector at that instant.*

That the statement is true for a dielectric as well as for a metal is

¹³ Such a wave could not exist in any physically possible medium, as it would require q to be a pure imaginary, which implies a dielectric constant zero. But as will appear shortly, this physical impossibility does not matter for our purpose.

obvious. *It is not true of elliptically polarized light*, which is the sum of two plane-polarized beams each of which is capable of separate existence. That, however, is of little moment. What is of consequence is that an equivalent statement can be made for either of the hybrid types.

Take first the case of Fig. 7(a) which applies to a dielectric only. Obviously the rotating H -vector is the resultant of two components, one in phase with E and lying along the major axis of the ellipse, the other 90° out of phase and lying along the minor axis. To the first corresponds a continuous energy transfer normal to both E and the major axis—that is, along the minor axis. To the other corresponds a surge and resurge in the direction of the major axis, *but with no transfer of energy in the long run.*

So far as average propagation is concerned, the conditions are again determined by the position of H when E is at maximum extension. As in the other cases, the energy travels, in the long run, in the direction of the phase velocity w . But there is this difference. The surging takes place *at right angles* to this direction, not along it; as if the energy traveled a sinuous path in the general direction of w , like a very drunken man walking down a perfectly straight street. Perhaps the reason the phase velocity of hybrid light is less than that of plane-polarized light is the same as the reason the very drunken man does not progress as fast as he goes.

In Fig. 8(a), where E rotates, the same general conditions exist: the average propagation is determined solely by the instantaneous position of the rotating vector (E) when the non-rotating vector (H) is at maximum extension.¹⁴

Finally, in case the hybrid beam is in a conductor, as in Fig. 7(b), the components of H along the major and minor axes are both out of phase with E . But an elliptical motion can be resolved in other ways than along the axes of the ellipse; it can be resolved along any pair of conjugate diameters. If we choose $\overline{PH}$ and its conjugate, the $\overline{PH}$ component will be in phase with E and the other 90° out of phase. Hence there is transfer of energy normal to both E and $\overline{PH}$ —which is again the direction of the phase velocity—and there is surge and resurge normally to E and the conjugate diameter—which is again the direction of the damping vector.¹⁵

Now in 8(b) the same argument applies in every respect but one:

¹⁴ Up to this point in fact, by the position of *either* when the other was at maximum extension, but this duality soon disappears.

¹⁵ The conjugate of $\overline{PH}$ is that line along which the H vector will lie a quarter-period later

the mean propagation is again normal to $\overline{PE}$, but the direction of this normal does not coincide with the phase velocity $\overline{PV}$. In other words, it appears to be a general law that

In plane-polarized or hybrid light, whether in dielectric or metal, the mean transfer of energy is, in both direction and magnitude, $c/8\pi$ times the vector product of the instantaneous E and H taken at the moment when the non-rotating component is at maximum extension: but it seems not to be true that this direction and the phase velocity always coincide. That is true in every case except hybrid light in a metal, with E rotating.

This one case *seems to be* unique in the sense that "the wave goes one way and the energy another." I say "seems to be" for the reason that, as I said at the beginning of this section, Poynting's theorem is attractive rather than true.

12. *The fundamental types of plane waves of light:* There is little need for a resumé to an expository paper. There are, however, several concepts upon which additional emphasis might be placed.

The "hybrid" waves are fundamental types in a sense in which circularly or elliptically polarized light are not. These latter can be made from—or made into—plane-polarized light; they are, basically, the same thing. Whether we think in terms of one or the other is a mere matter of convenience. But hybrid waves cannot be made from plane-polarized waves; nor the two types from one another.¹⁶

Moreover, as we have said, within a metal the "absorbed light" is always of one of these types, except in the extremely special case of normal incidence. It is obvious that in studies of energy absorption and re-emission, whether as photo-electricity or radiation, the fact that one vector is *always* rotating, and both *sometimes*, should be kept in mind.

Finally, I may remark that our use of Cartesian coordinates is really an artificial limitation. Substantially the same sort of argumentation, which is all based upon discussing what happens when certain constants become complex, is possible in any frame of reference; and in the case of cylindrical co-ordinates, at least, it leads to some very curious results, which I may perhaps present at a later time.

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It has a phase of $\eta - \pi/2$, while $\overline{PH}$ has the phase η . The damping and velocity vectors of 7(b) are "conjugate."

¹⁶ It may be that they can be built up by the superposition of an infinity of plane-polarized waves, Fourier integral fashion. The statement refers only to decomposition into two simpler types.

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PLANE WAVES OF LIGHT--II

BY

THORNTON C. FRY

A PRESENTATION OF THE
EQUATIONS GOVERNING REFLECTION AND REFRACTION,
WITH PARTICULAR ATTENTION TO THE PRODUCTION
OF HYBRID WAVES

REPRINT OF PAPER

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PLANE WAVES OF LIGHT

II. REFLECTION AND REFRACTION

By THORNTON C. FRY

§13. *Introduction:* In a previous paper¹ the propagation of plane light waves through a homogeneous isotropic medium was considered from the standpoint of Maxwell's theory. It was shown, principally, that there are two *essentially* different types of plane waves, the one being plane-polarized light, and the other a type which, for lack of a recognized name, we called "hybrid." All other waves that could be represented by an exponential of the type

$$e^{iq(\alpha x + \beta y + \gamma z) - ipt}$$

could be produced by the superposition of plane-polarized or hybrid waves: but neither of these could be derived from the superposition of waves of the other type. In that paper certain references were made to reflection and refraction: for instance, it was stated that the electromagnetic disturbance on the "dark" side of a totally reflecting glass boundary is "hybrid." But these statements were made without proof, and nowhere in the paper was the problem of two media discussed.

The present paper aims to present the equations governing reflection and refraction, and to show how they lead in metals to hybrid waves. As in the earlier paper, many of the results are well known, but I have failed to find them collected in one place in a systematic manner. Besides, the classical notation for reflection and refraction proves to be a very awkward medium of expression when angles of refraction become "imaginary," as they do in dealing with metals, and it appears desirable

¹ Plane Waves of Light: I. Electromagnetic Behavior, J.O.S.A. & R.S.I., 15, p. 137; 1927.

to have a symbolism better suited to our purpose. It is for this reason that a certain amount of old material masquerades in a new dress in succeeding sections.

§14. *The Electromagnetic Boundary Conditions:* Why do we associate with an incident beam *two* others: a reflected one and a refracted one? And why do we say these are of the same frequency as the incident beam? One answer, of course, is that we observe this to be true; but in discussing such a matter as reflection from a purely theoretical standpoint such as that afforded by Maxwell's Theory it is interesting to see how far our theoretical hypotheses alone would carry us. With reference to these specific questions we shall find they carry us exactly as far as the experimental evidence goes, and (so long as we deal with dielectrics) no further.

To see this, we may recall that the same physical ideas which lead to Maxwell's equations also show that if two media are in contact, their electromagnetic fields must be related in a very definite fashion. In other words, they give, not only the equations themselves, but certain boundary conditions as well. These conditions are:

- I. The normal component of μH must be continuous.
- II. The tangential components of E must be continuous.
- III. The normal components of ϵE must differ by an amount equal to the surface charge. That is, if there is a charge s per unit area, and if the normal to the surface is called z , the values on the positive and negative sides must satisfy the relation $\epsilon_+ E_{z+} - \epsilon_- E_{z-} = s$.
- IV. The tangential components of H differ by amounts depending on the currents flowing in the surface. Taking the direction of z normal to the surface, and x and y tangential to it, the relations are

$$H_{y+} - H_{y-} = -I_x/c,$$

$$H_{x+} - H_{x-} = +I_y/c,$$

I_x and I_y denoting the currents flowing across unit lengths parallel to the x and y axes, respectively.

In the case of dielectrics, there is no occasion to expect the existence of surface charges or currents. Hence all four vector-components are continuous. But in the case of metals, conduction currents do give rise to surface charges, and probably also to surface current sheets.² The

² These sheets are of speculative character, since they depend upon the behavior of the free surface electricity under the influence of the electric field. It appears possible, however, that they may introduce very weak light of *twice the incident frequency*, which would be interesting, if true.

latter, however, if they exist at all, are so feeble that they may be ignored in this discussion. Hence we have *three* general conditions:

The normal component of μH , and the tangential components of E and H , are all continuous.

Obviously, then, if a wave of light is moving through the first medium, some sort of disturbance must exist in the second, and the mathematical problem presents itself of determining the character of the latter disturbance. But, of course, to do that presupposes that we know the nature of the disturbance in the first medium. In our study we shall assume it to be a wave of the exponential form³

$$\begin{aligned} E_x &= E l e^{i q (a x + b y + c z) - i p t}, \\ &\dots \dots \dots \\ H_z &= H n' e^{i q (a x + b y + c z) - i p t}. \end{aligned} \quad (2)$$

That is, it is plane, or elliptically polarized, or hybrid.

The simplest guess which we can make regarding the nature of the disturbance in the second medium is, that it is of the same nature as that in the first; that is, that it is of the form

$$\begin{aligned} E_{x_1} &= E_1 l_1 e^{i q_1 (a_1 x + b_1 y + c_1 z) - i p_1 t}, \\ &\dots \dots \dots \\ H_{z_1} &= H_1 n'_1 e^{i q_1 (a_1 x + b_1 y + c_1 z) - i p_1 t}. \end{aligned}$$

If so, at the bounding surface (which we assume to be the plane $z=0$) E_{x_1} , E_{y_1} , H_{x_1} , H_{y_1} , $\epsilon_1 E_{z_1}$ and $\mu_1 H_{z_1}$ must equal the corresponding quantities without the subscripts.

We take first the relation between E_x and E_{x_1} . It is

$$E l e^{i q (a x + b y) - i p t} = E_1 l_1 e^{i q_1 (a_1 x + b_1 y) - i p_1 t}.$$

This relation must be true at every instant and at every point of the surface, which it will be only provided

$$\begin{aligned} p &= p_1, \\ q a &= q_1 a_1, \\ q b &= q_1 b_1. \end{aligned}$$

Physically, these equations state that the light is of the same frequency in both media, and that the traces made upon the interface by waves in either medium move with like velocities.

³ Equations are numbered as in the previous paper.

Moreover, q_1 is uniquely determined by the properties of the medium, and therefore these equations completely determine a_1 and b_1 . Then from the equation

$$a_1^2 + b_1^2 + c_1^2 = 1, \quad (3)$$

the triplet a_1, b_1, c_1 can be *completely* determined, except that c_1 is obtained by extracting a square root, and may be either positive or negative. This triplet determines the direction of propagation and the damping of the wave. Hence:

If it is possible to satisfy the boundary conditions at all by a wave of this form, it must move in one of two directions.

Moreover, whether these waves are of the plane-polarized or hybrid type, and in the latter event the direction and degree of the damping, are completely determined.

This latter statement is justified by the fact that, as was seen in the earlier paper, the nature of hybrid light is entirely determined by the triplet (a, b, c) .

We must add that all this is quite independent of our assumption that no surface charges exist. The presence of such surface conditions may modify the plane of polarization or the intensity of the wave. They cannot affect its type. This is obvious from the fact that we have used only condition II, which is universally valid.

There are, then, just two sorts of waves which may exist in the second medium. Presumably either of these waves might exist alone, or both might exist together, and our next step must be to investigate whether the boundary conditions can be satisfied by either of these three possibilities. We need not carry out the mathematics here: the result is, that one beam alone can exist only provided a is given a special value; that is, provided the light in the first medium is incident at a particular angle, which is known as the polarizing angle. But the two together can satisfy the boundary values, no matter what the beam in the first medium may be.

Generally, therefore, there are two waves in one of our media. Mathematically, we have thought of these as arising from the known disturbance in the other medium: physically we ordinarily invert the roles, regard one of the pair (the "incident beam") as known, and say that it gives rise to both its mate (the "reflected beam") and the disturbance in the other medium (the "refracted beam"). Both are ways of thinking: the logical fact is that the three coexist. But from this point on we shall find the physical way the better.

Finally, we must not forget that all this has been achieved by *assuming* that all disturbances were of exponential type; and we must now inquire whether any *other* solution is possible. It is not difficult to see that if there is such a solution, it must be equivalent to superposing upon the waves already found, some disturbance which vanishes *over the entire boundary and at all times*. As such a disturbance could exist without any of the waves with which we are dealing, it cannot be said to be "produced by" any of them.⁴ Hence, if such solutions exist at all, they are outside the range of our physical problem.

So much for pure logic. We now return to algebra.

§15. *The Equations for Simple Reflection at a Plane Boundary:* We have justified our belief in two facts: (a) That a wave of exponential form "gives rise to" waves of exponential form, and to such waves only, when it is allowed to impinge on a plane boundary. (b) That the boundary conditions can only be satisfied by assuming *three* waves: the "incident" and "reflected" beams in one medium, the "refracted" beam in the other. This leaves us, then, with only the problem of determining the exact relationships which exist between the various constants that characterize the three waves.

It will serve our purpose best to assume that the incident light is plane-polarized, with either the electric or the magnetic vector in the plane of incidence, for this will permit us to choose our coordinate axes in a very convenient fashion. Moreover, the loss in generality is not serious, for the incident beam is seldom hybrid in practice, and all other cases can be built up by superposing waves of the types discussed.

We choose our axes in such a way that the bounding plane contains x and y , and the plane of incidence x and z . We also agree to denote by the subscript 1 those quantities which appertain to the refracted beam, by 2 those associated with the reflected beam, while those associated with the incident beam are without subscripts. Finally, we adopt the letter r to denote the sine of the angle of incidence. Then the triplet a, b, c automatically takes the form $r, 0, -(1-r^2)^{1/2}$; while from the argument of §14 we conclude that

$$q_1 a_1 = qa,$$

$$q_1 b_1 = qb,$$

$$a_2 = a,$$

$$b_2 = b;$$

⁴ Physically this condition exists when polychromatic light is allowed to fall upon the surface—each color individually is reflected without regard to the presence of the others.

or, in tabular form:

$$\begin{array}{lll}
 a=r, & a_1=(q/q_1)r, & a_2=r, \\
 b=0, & b_1=0, & b_2=0, \\
 c=-(1-r^2)^{1/2}, & c_1=\pm(1-q^2r^2/q_1^2)^{1/2}, & c_2=+(1-r^2)^{1/2}.
 \end{array} \quad (52)$$

If a and c are real, as we are assuming, a_2 and c_2 are also. Hence,

If the incident light is not of hybrid type, the reflected light is not, no matter whether the surface from which reflection occurs is that of a metal or a dielectric. On the other hand, if one medium is a dielectric and the other a metal, one q is real and the other (say q_1) is complex. Hence q/q_1 is not real, whence a_1 and c_1 must be complex, and the refracted light hybrid. The sole exception is normal incidence, for which $r=0$. Hence

Plane-polarized light refracted through the boundary between a dielectric and a metal becomes hybrid, except at normal incidence.

As hybrid light travels more slowly than plane-polarized light, this raises some doubt as to the possibility of determining the refractive index of metals by direct measurement. These will be considered in detail in §20.

This, however, is not the only way of producing hybrid light; for it is possible for q^2r^2/q_1^2 to exceed unity even though both q and q_1 are real, provided $q > q_1$ (that is, provided the first medium is optically denser than the other). Then, though both media are dielectrics, c_1 will be complex, and the refracted beam will be hybrid. We shall find that in such cases there is no propagation of energy into the second medium, wherefore the incident wave is said to be "totally reflected."

There is an essential difference in the character of these two cases, obviously, and we should expect this difference to manifest itself somehow in the nature of the triplet a_1, b_1, c_1 . It does so in the fact that, when a *metal* is involved (so that q/q_1 is complex) both a_1 and c_1 are complex; while when both media are dielectrics, a_1 is real and c_1 pure imaginary. This we may indicate by writing

$$\begin{aligned}
 q_1 a_1 &= qa + i0, \\
 q_1 b_1 &= 0 + i0, \\
 q_1 c_1 &= R + iI,
 \end{aligned}$$

in the one case, and

$$\begin{aligned}
 q_1 a_1 &= qa + i0, \\
 q_1 b_1 &= 0 + i0, \\
 q_1 c_1 &= 0 + iI,
 \end{aligned}$$

in the other. It takes only a cursory inspection of these sets to show that in the second case the damping vector is in the z -direction and the velocity vector in the x -direction, as in Fig. 9 (a), while in the first case the velocity is inclined to the system of coordinate axes, as in Fig. 9 (b).

Now we have shown in the first paper that energy always flows in the direction of the velocity vector.⁵ Hence in case (a) there is no flow from the first medium to the second.⁶ The energy surges and resurges across the boundary, but gets nowhere in the end.

In the other case, part, at least, of the energy which crosses the boundary never returns, but is conducted away. Until we have evaluated the amplitude factors E_1 and H_1 we cannot be assured that this

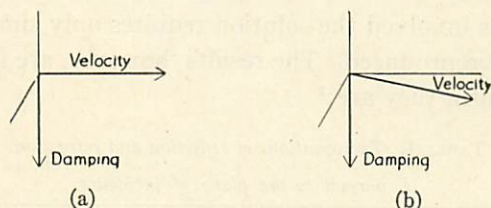


FIG. 9

really means a transport of energy; for if either happened to come out zero there would still be no loss. As we shall see, this latter state of affairs would occur in the case of a perfect conductor, for which E_1 is zero. It, too, is a perfect mirror, not because all the energy which passes it is returned, but because none ever passes.

So much for the triplets a, b, c . Next we wish to evaluate the other triplets and the amplitudes E_1 and H_1 . In doing this it is convenient to confine our attention to two cases, one characterized by the fact that E is normal to the plane of incidence, the other by the fact that H is normal to that plane. So long as the incident beam is not hybrid all possible cases can be built up by the superposition of waves of this sort. The typical cases are obviously characterized by the triplets

⁵ Except in one case. In that case, however, the Poynting vector has the same component as the velocity *normal* to the bounding surface: the two differ only in their *tangential* components. Hence our present argument is not affected.

⁶ The question is sometimes raised: If no net energy flows across the surface, how can there be any energy in the lower medium? The answer is, that the statement only applies to the steady-state condition. When light is first caused to shine on the surface, there is for a time a net flow into the metal; when the light is shut off, a net flow out of it. These, however, are transient effects. During the intermediate period, with equilibrium established, no net flow takes place.

$$\left. \begin{array}{l} l=0 \\ m=1 \\ n=0 \end{array} \right\} (E \text{ normal to plane of incidence}),$$

$$\left. \begin{array}{l} l'=0 \\ m'=1 \\ n'=0 \end{array} \right\} (H \text{ normal to plane of incidence}),$$

and must be separately considered.

The algebraic process is very simple: We start with one of these triplets, together with the known triplets (52), substitute them in the fundamental equations (3) to (8) of the previous paper, and then apply the three boundary conditions built up in §14. As only the simplest sort of algebra is involved the solution requires only time and patience, and need not be reproduced. The results, however, are important. Listed in tabular form they are:⁷

TABLE 1. *The equations of reflection and refraction.*

<i>E normal to the plane of incidence</i>		
$l=0$	$a=r$	$l'=-c$
$m=1$	$b=0$	$m'=0$
$n=0$	$c=-(1-r^2)^{1/2}$	$n'=a$
$l_1=0$	$a_1=q_1r/q_1$	$l'_1=-c_1$
$m_1=1$	$b_1=0$	$m'_1=0$
$n_1=0$	$c_1=\pm(1-q_1^2r^2/q_1^2)^{1/2}$	$n'_1=a_1$
$l_2=0$	$a_2=r$	$l'_2=-c_2$
$m_2=1$	$b_2=0$	$m'_2=0$
$n_2=0$	$c_2=\pm(1-r^2)^{1/2}$	$n'_2=a$
$\frac{E_1}{E} = \frac{k-k_2}{k_1-k_2} \qquad \frac{E_2}{E} = -\frac{k_1-k}{k_1-k_2}$		

Notes:

1. The symbol " k " is used for the combination iqc/h , the q , c and h all being understood to have the same subscripts as the k .

2. It is not necessary to write equations for the H 's; they are sufficiently defined by means of (9).

3. The ambiguous signs are to be so chosen that the real part of c_1 is negative and the real part of c_2 positive. The significance of other combinations is discussed in the following paragraphs.

⁷ I need hardly say that the ordinary laws of reflection and refraction are all contained in this table—not implicitly but *explicitly*. If they are not immediately obvious, it is only because the notation is unfamiliar.

TABLE 2. *The equations of reflection and refraction.**H normal to the plane of incidence*

$l=c$	$a=r$	$l'=0$
$m=0$	$b=0$	$m'=1$
$n=-a$	$c=-(1-r^2)^{1/2}$	$n'=0$
$l_1=c_1$	$a_1=q_1r/q_1$	$l'_1=0$
$m_1=0$	$b_1=0$	$m'_1=1$
$n_1=-a_1$	$c_1=\pm(1-q_1^2r^2/q_1^2)^{1/2}$	$n'_1=0$
$l_2=c_2$	$a_2=r$	$l'_2=0$
$m_2=0$	$b_2=0$	$m'_2=1$
$n_2=-a_2$	$c_2=\pm(1-r^2)^{1/2}$	$n'_2=0$
$\frac{H_1}{H} = \frac{\kappa - \kappa_2}{\kappa_1 - \kappa_2}$		$\frac{H_2}{H} = -\frac{\kappa_1 - \kappa}{\kappa_1 - \kappa_2}$

Notes:

1. Here κ is used for iqc/g .
2. The E 's are understood to be defined by (9).
3. The real part of c_1 is negative; the real part of c_2 positive.

Every quantity used in discussing our waves is completely determined in these tables, except that the signs of c_1 and c_2 are ambiguous. There is no *mathematical* way of deciding which signs to use, any combination being a solution of the problem. We must therefore revert to the physical meanings of the quantities for their further definition. This can best be done by thinking, for a moment, of a case in which a_1 and c_1 , as well as a_2 and c_2 , are real. They then represent direction cosines of the velocity of propagation; and in particular c represents the

For example, the usual law of sines is contained in the equation

$$a_1 = \frac{q}{q_1}r,$$

or

$$\frac{a_1}{r} = \frac{q}{q_1};$$

for when the media are dielectrics the velocities in the two media are $v=p/q$ and $v_1=p/q_1$, as was said in § 4. Moreover, in this case a_1 is the sine of the angle of refraction, and r the sine of the angle of incidence. Hence $a_1/r=q/q_1$ becomes at once $\sin R/\sin I=v_1/v$.

Similarly the amplitude relations, when rewritten in trigonometric notation, are just the usual equations.

The polarizing angle is that angle for which $H_2=0$. From Table 2 it is easily seen to be given by $\kappa=\kappa_1$, which easily reduces to $\cos I/\cos R=v_1/v$, and this, together with the law of sines leads at once to the more familiar relation $\tan I=v_1/v$.

Other common properties appear with equal ease.

vertical component of velocity. Then when c is positive the wave moves upward, and when c is negative it moves downward. Keeping this idea in mind, the four possible cases are seen to be represented schematically by the diagrams of Fig. 10. They may be characterized in words as follows: (a) Two waves, 180° out of phase, superposed. They are equally intense, and therefore interfere destructively. The refracted beam is of zero intensity. The entire case is trivial, but physically possible. (b) The usual case of reflection. (c) Identical with (a), except for the direction of the refracted beam. As this is of zero intensity its direction is obviously inconsequential. Trivial, but physically possible. (d) If a beam were caused to impinge upon the bounding surface *from beneath*, it too would have "reflected" and "refracted" associates. By properly adjusting the angle of incidence, they can be caused to coincide with the beams refracted and reflected from above, respectively.

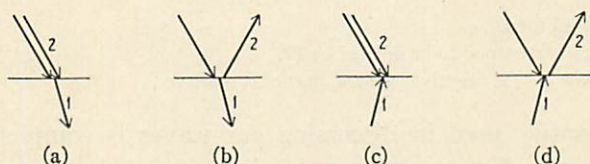


FIG. 10

By properly adjusting the amplitude and phase of the beam incident from below, its reflected beam can be caused to destroy the beam refracted from above. This is the condition of (d). It is physically trivial, but we shall find it possible to build a useful mathematical artifice upon it later on.

If c_1 or c_2 is complex the same set of possibilities is met, except that for " c positive" we must read "the real part of c positive"; and so on.

§16. *Two Numerical Illustrations:* We conclude this part of our study with two numerical illustrations, one of non-metallic reflection, the other of metallic reflection.

For the non-metallic case, we choose glass with an index of refraction 1.5 as our upper medium, and air as the lower one. Then the history of a light wave, as its angle varies from 0° to 90° (in the glass) is told by Fig. 11. The electric vector is supposed to be normal to the plane of incidence.

The upper third of the figure is devoted to direction of propagation, each set of three vectors representing corresponding velocities of incident, reflected and refracted light. As we pass across the figure from left to right the angle of incidence increases, passing from normal

incidence through the angle of total refraction (polarizing angle) and the angle of total reflection to grazing incidence. Until the angle of total reflection is reached, the angle of refraction increases more rapidly

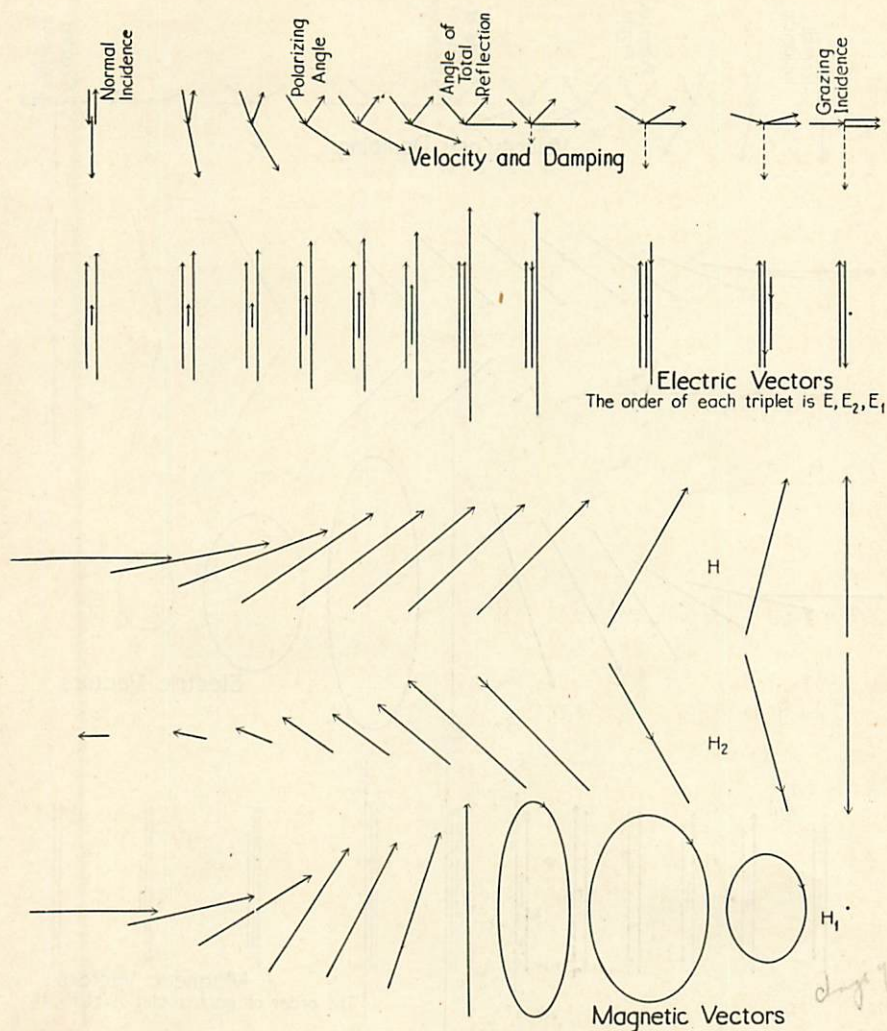


FIG. 11. Reflection at a glass-air boundary: electric vector perpendicular to the plane of incidence.

The dotted lines in the upper part of the figure represent the damping vector.

than the angle of incidence and the velocity of the refracted beam remains unaltered; thereafter the angle remains fixed at 90° while the velocity decreases, as shown by the decreasing length of the vector.

At 60° incidence, for example, the refracted beam is moving with only 77% of the usual velocity of light in air. As the velocity decreases, however, a "damping" vector normal to the surface makes its ap-

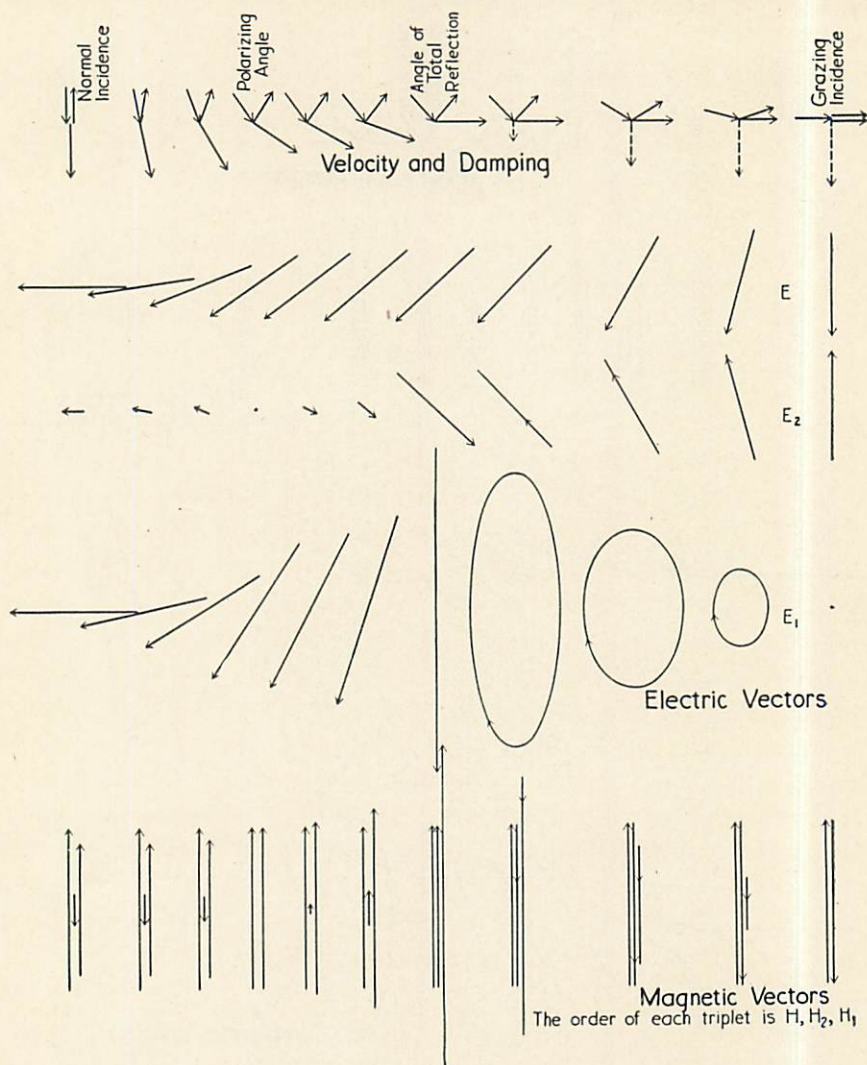


FIG. 12. Reflection at a glass-air boundary: magnetic vector normal to the plane of incidence. The dotted lines in the upper part of the figure represent the damping vector.

pearance, and rapidly increases in magnitude. It is shown by the dotted lines.

The scale is so chosen that unit length of the velocity vector represents the velocity of light *in vacuo*, while the damping vector gives the "extinction coefficient" directly.

The second section of the figure is devoted to the E -vectors, the top ends of which must be supposed to point away from the reader if the velocity and H are in the plane of the paper. The amplitude of the reflected light increases until the angle of total reflection is reached.

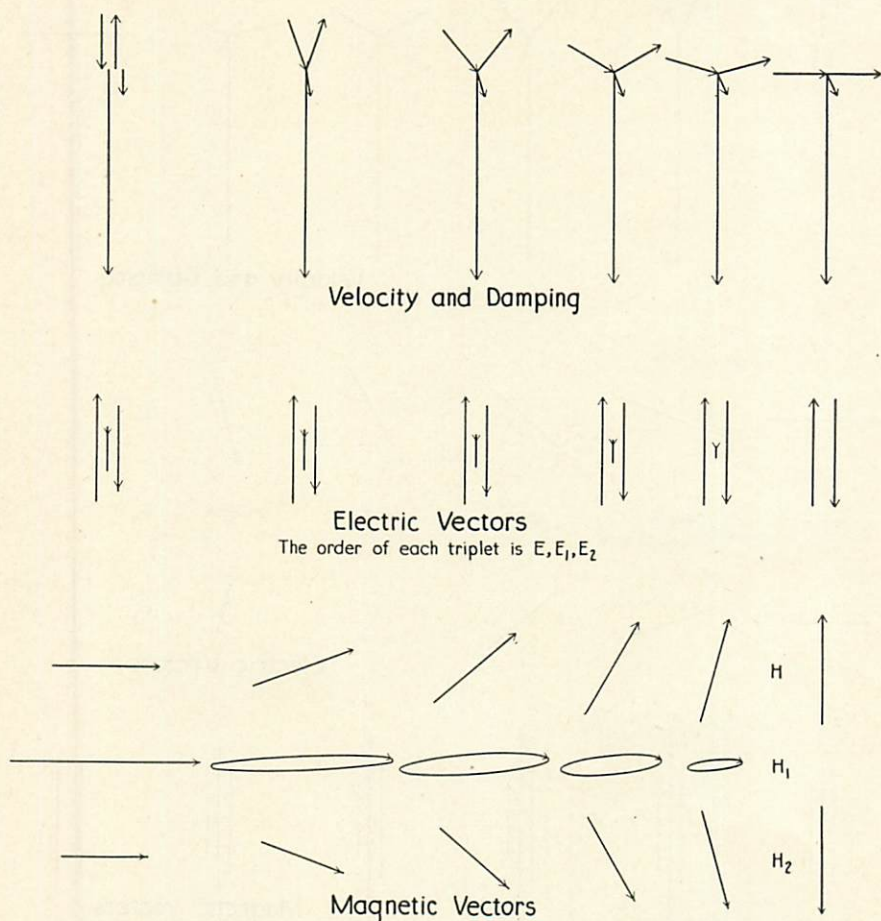


FIG. 13. Reflection at an air-platinum boundary: electric vector normal to the plane of incidence. In the top part of the figure the vertical vectors, which do not change direction as the angle of incidence changes, represent the damping.

From this point on it remains constantly equal to the incident vector, but becomes progressively more and more out of phase with it. The refracted vector increases up to the angle of total reflection; then decreases steadily, with constantly changing phase, to zero at grazing incidence.

The remainder of the figure is devoted to the H vectors, which behave in a perfectly familiar fashion until the angle of total reflection is reached. Thereafter, in the case of the refracted light, they describe ellipses in the plane of the paper. As the angle of incidence increases,

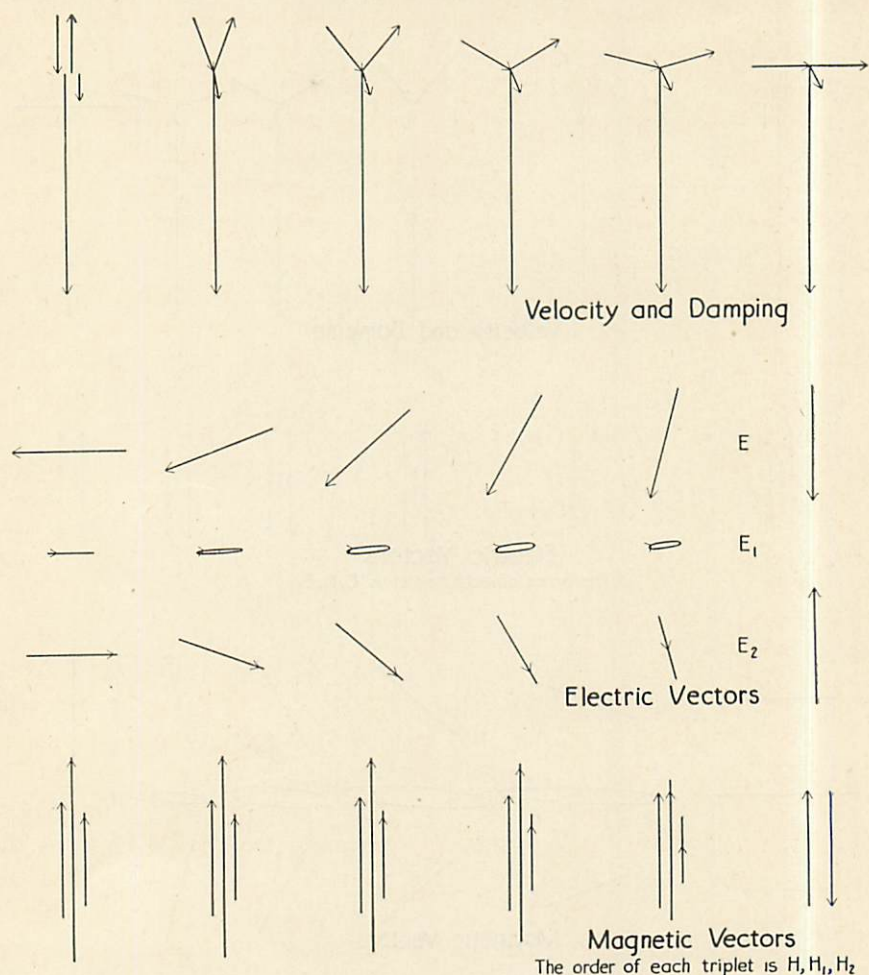


FIG. 14. Reflection at an air-platinum boundary: magnetic vector normal to the plane of incidence.

In the top part of the figure the vertical vectors, which do not change direction as the angle of incidence changes, represent the damping.

these ellipses become shorter and fatter and vanish into a point at grazing incidence.

Fig. 12 is similar to Fig. 11, except that H instead of E is supposed normal to the plane of incidence. It needs no comment, except to say

that the tops of the H -graphs must point away from the reader; the others lie in the plane of the paper.

As an illustration of metallic reflection, we may choose platinum for our mirror, and allow the light to impinge upon it from above. The life history is given graphically in Figs. 13 and 14. The principal points to be noted are:⁸

1. The velocity and damping vectors are no longer at right angles. (The damping vectors are the long ones, which point vertically downward. The short ones, which change direction with changing angle of incidence, are the velocity vectors.) There is therefore a component of damping in the direction of propagation, which implies dissipation of energy.

2. The velocity and damping change but little with increasing angle of incidence. In the case of non-metallic total reflection the change was marked.

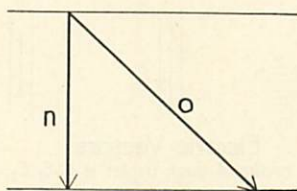


FIG. 15

This has a peculiar consequence, which I suspect has never been noticed. It is, that light which impinges *obliquely* upon a conductor will travel further in it than will light which impinges normally. This is readily seen from Fig. 15; for if the damping is normal to the surface and substantially independent of the angle of incidence, oblique light must travel the distance o to suffer the same attenuation that normal light suffers in a distance n .

This is not in contradiction to the fact observed in the case of semi-transparent liquids, such as dye-stuffs, that the attenuation is proportional to the distance travelled, whatever the angle of incidence; for the conductivity of such substances is low, and the theory then predicts a damping approximately proportional to the cosine of the refraction angle. For substances of *high* conductivity, however, the light penetrates to substantially the same *depth* (not *distance*) regardless of the angle of incidence.⁹

⁸ The figures are drawn for a wave length 4346A, the optical constants being taken as $N=2.10$, $K_o=3.80$.

⁹ It need hardly be said that the *intensity* at a given depth is not independent of the angle of incidence; for that intensity depends upon *how much light enters* the conductor, as well as

3. Perhaps the most amazing observation is with regard to the behavior of the rotating vectors. In the case of Fig. 13, for example, it is entirely natural for the vector H_1 to be horizontal when light is incident normally; but as the angle of incidence increases and the in-

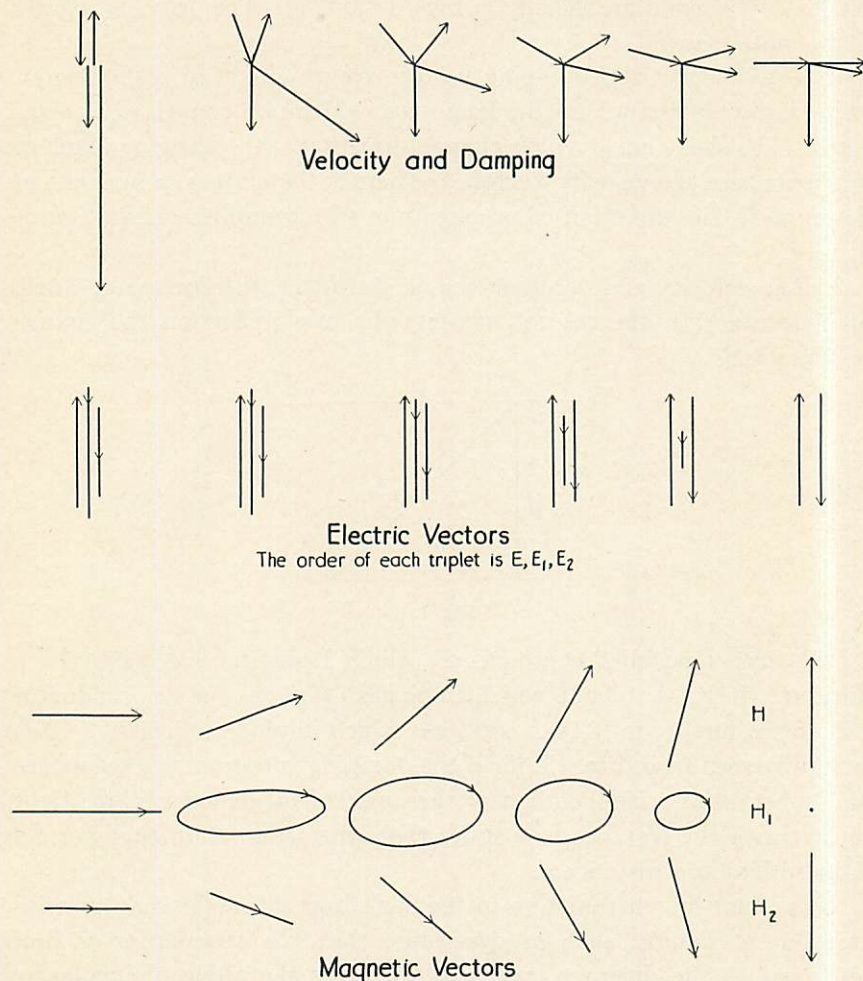


FIG. 16. Reflection at an air-rubidium boundary: electric vector normal to the plane of incidence.

In the top part of the figure the vertical vectors, which do not change direction as the angle of incidence changes, represent the damping.

cident H -vector cuts the surface at a greater and greater angle, we should expect the refracted H_1 to tilt upward to a marked degree.

upon how rapidly it is absorbed. The former quantity depends markedly upon the angle of incidence; the latter only to a very small extent.

Such a tilt is apparent in the figure, of course, but not to the extent that one would expect. For 75° incidence, for example, the refracted beam is travelling at an angle of 25° from the vertical, yet the long dimension

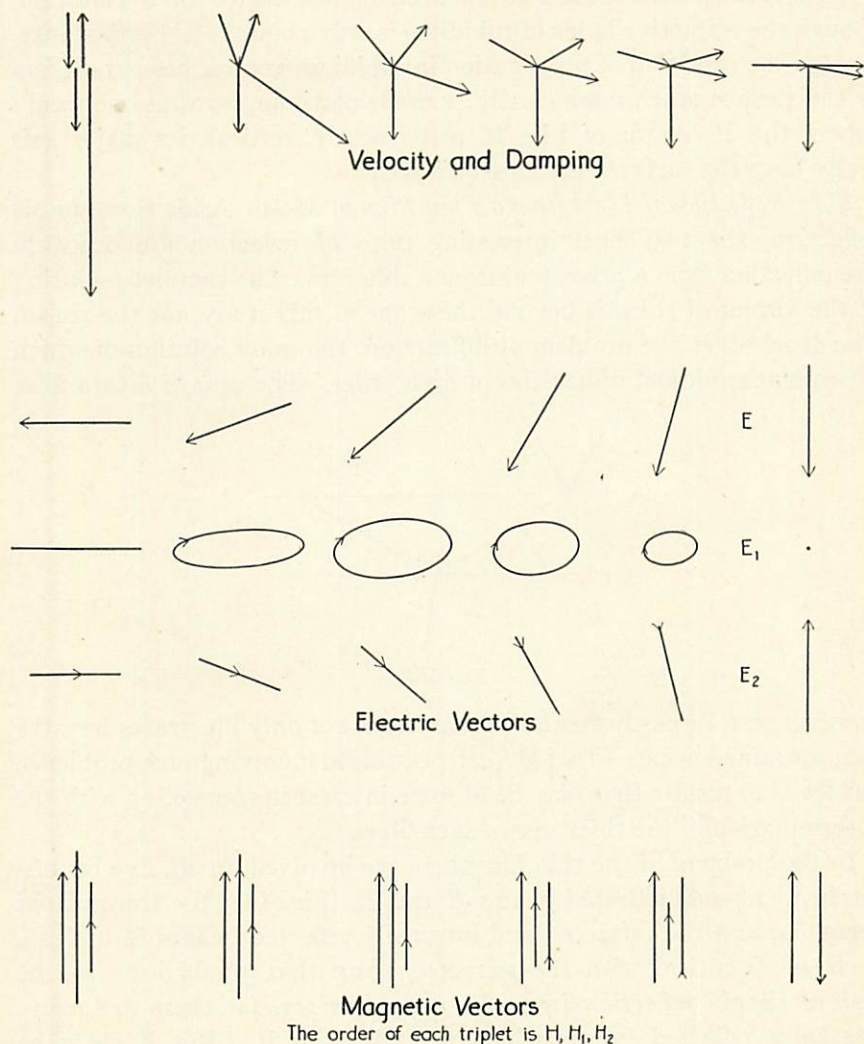


FIG. 17. Reflection at an air-rubidium surface: magnetic vector normal to the plane of incidence. In the top part of the figure the vertical vectors, which do not change direction as the angle of incidence changes, represent the damping.

of the H_1 -ellipse departs only 6° from the horizontal. It appears to be a property of metals to keep the H and E vectors as nearly as possible parallel to the surface no matter what the angle of incidence may be.

4. Finally, the statements made above are equally as true for metals of low refractive index as for platinum, the refractive index of which is about 2. Thus Figs. 16 and 17, which are drawn for rubidium, exhibit the peculiarities mentioned to an even higher degree than platinum, though the refractive index of rubidium is only about $1/4$. For instance, though the direction of propagation in rubidium approaches parallelism to the surface much more closely than for platinum, so that one would expect the H_1 -vector of Fig. 16 to be nearly vertical, its major axis really hugs the surface very closely indeed.

§17. *Reflection of Light from a Thin Film of Metal.* Aside from simple reflection, the two most interesting cases of reflection are probably the reflection from a prism and from a thin film. The complete solution of the former of these is beyond the scope of this study, for the reason that it involves the problem of diffraction, the exact solution of which raises mathematical difficulties of high order. The case of a thin film,

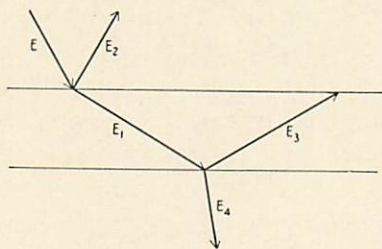


FIG. 18.

however, can be easily treated; moreover it not only illustrates how the idea contained in case (d) of §15 can be utilized in solving such problems, but leads to results that may be of some interest in connection with the determination of the thickness of such films.

In the problem of the thin film there are involved, in all, five beams: the incident and reflected beams E and E_2 (Fig. 18); the transmitted beam E_4 , and the refracted and internally reflected beams E_1 and E_3 . E_1 is *not* identical with the refracted beam that would exist in the case of simple reflection from the upper surface, for there are many internally reflected beams which coincide with it. But it *could* be produced by simple reflection, if the proper incident intensity were produced, and if there were no internal reflection. Suppose the proper incident intensity were E_5 , and the resulting reflected intensity E_6 (Fig. 19).

The internally reflected beam E_3 , also, *could* be associated with an incident beam E_7 , and a reflected beam E_8 after the manner of Fig. 10

(d), and this combination would have no component in the direction of E_1 . Moreover, these two possible arrangements *could* be superposed: if so, the resultant disturbance in the second medium would be identi-

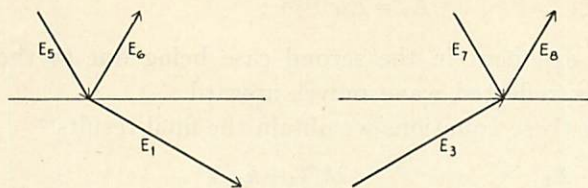


FIG. 19.

cally the same as in the actual case of the film, so the resultant disturbance in the first medium must also be just the same. That is,

$$E = E_5 + E_7,$$

$$E_2 = E_6 + E_8.$$

Finally let us turn our attention to the lower surface. When the resultant refracted beam reaches this point, it no longer has its original amplitude E but has suffered phase, and possibly intensity, change. Let us call its new amplitude E_1' , and let us agree that a prime affixed to any other quantity shall, for the moment, designate its value at the lower surface. Then a moment's consideration shows that E_1' , E_3' and E_4' are related to one another by the conditions of simple reflection.

From these relations we obtain the equations

$$E_4'/E_1' = 2k_1/(k_4 + k_1),$$

$$E_3'/E_1' = -(k_4 - k_1)/(k_4 + k_1);$$

$$E_1/E_5 = 2k/(k_1 + k),$$

$$E_6/E_5 = -(k_1 - k)/(k_1 + k);$$

$$E_3/E_7 = 2k/(-k_1 + k),$$

$$E_8/E_7 = (k_1 + k)/(-k_1 + k).$$

Finally we must find the relation between the corresponding primed and unprimed quantities. From (2), which is the general law of propagation for our wave, we have

$$E e^{i q (a x + b y) - i p t}$$

as the equation of the disturbance at $z=0$; and

$$E e^{-i q \zeta} e^{i q (a x + b y) - i p t}$$

as its equation at the surface $z = -\zeta$, ζ being, of course, the thickness of the film. These differ only in that the amplitude factor E has been

replaced by the new amplitude factor $E' = Ee^{-iqc\tau}$. Hence we have at once

$$\begin{aligned} E_1' &= E_1 e^{-iq_1 c_1 \tau}, \\ E_3' &= E_3 e^{iq_1 c_1 \tau}; \end{aligned}$$

the positive exponent in the second case being due to the fact that the internally reflected wave moves upward.

By solving these equations we obtain the final results¹⁰

$$\begin{aligned} \frac{E_1}{E} &= \frac{2k(k_1 + k_4)X^2}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \\ \frac{E_2}{E} &= \frac{(k - k_1)(k_1 + k_4)X^2 + (k + k_1)(k_1 - k_4)}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \\ \frac{E_3}{E} &= \frac{2k(k_1 - k_4)}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \\ \frac{E_4'}{E} &= \frac{4kk_1X}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \end{aligned} \tag{53}$$

where

$$X = e^{iq_1 c_1 \tau}.$$

These equations are written in a form corresponding to the state of polarization in which E is normal to the plane of incidence. To deal with the other case it is only necessary to replace k by κ , and E by H .

When computations are made from these equations they reveal certain peculiarities of reflection from metallic films which are worth noticing in passing.

In the first place, there is a certain thickness which absorbs more energy than does an infinitely thick block of metal, though the film in question is still comparatively transparent. The effect is, of course, an interference phenomenon of the "resonance" type.

In the second place, films which are still so thin as to be virtually transparent exhibit marked selectivity as to plane of polarization, particularly at high angles of incidence, with the result that the state of polarization of the reflected light differs materially from that of the incident light. Since such changes in polarization are easily measured by means of a Babinet compensator, the idea at once presents itself of making them the basis for determining the thickness of such films.

¹⁰The same results can, of course, be obtained by summing the infinite series of internally reflected components.

With this idea in mind, special attention has been given to the effect of this "selective absorption" upon the reflected light, and some very interesting experiments have been carried out to determine the thickness of "invisible" films of rubidium on glass and platinum. The results of these experiments will be found in a paper by H. E. Ives.¹¹ Their consistency is probably the best argument in support of the use of a continuous medium theory in dealing with matter which is decidedly "granular." For the purposes of this paper, therefore, we disregard this question entirely, and investigate only the mathematical aspects of the problem.

§18. *The Measurement of the Thickness of Thin Films*: Light polarized at an angle of 45° to the plane of incidence is, effectively, composed of two beams of equal intensity, one with the electric vector *in* the plane of incidence, the other with the electric vector *normal* to that plane. We can, if we like, think of each of these components as producing its own reflected beam: and it is quite obvious that we can easily compute the relative intensities and phases of these reflected components by the use of equations (53). To do so requires some tedious arithmetic but nothing more.

The result we get, of course, will depend upon the thickness of film for which the computations are made, and we could, if we so desired, carry out a set of such computations for various thicknesses.

We could also measure the same quantities by means of a Babinet compensator and its associated Nicol prism, the first of which measures the phase difference directly, and the second the angle θ defined by the equation

$$\tan \theta = \frac{\text{component } E \text{ normal to the plane of incidence}}{\text{component of } E \text{ in the plane of incidence}}.$$

Then, by comparing the *measured* results with the *computed* ones, we could judge the thickness of the film.

Such computations have actually been carried out, in the case of rubidium films deposited on glass, and have led to the curves shown in Figs. 2 and 3 of the paper already referred to.

¹¹ "The Thickness of Spontaneously Deposited Photoelectrically Active Rubidium Films, Measured Optically," J.O.S.A. & R.S.I., 15, p. 374; 1927. I take this opportunity to acknowledge my indebtedness to Dr. Ives in connection with this study of Plane Waves of Light. Not only did it have its origin in the attempt to answer certain specific questions which he raised, but it has had throughout the encouragement of his continued interest and the benefit of his knowledge of experimental optics, concerning which I was (and by comparison still am) in a state of profound ignorance.

It was found that the method was a very sensitive one when the optical properties of the film differed materially from those of the backing plate, as was the case with a rubidium film on glass, though with rubidium on a platinum background it did not work so well.

§19. *The Measurement of Optical Constants:* Among the thicknesses of "film" considered in the study to which we have just referred was infinity—which corresponds, of course to bulk material, and in this case the agreement between experiment and computation, as portrayed in Dr. Ives's paper, is excellent. If, however, incorrect optical constants had been used in the computations the agreement would have been poor. In other words, were the optical constants of the metal not known, they could be determined by the requirement that the curves computed from them must agree with the experimentally determined points. This is, in fact, the method usually used for the determination of optical constants of metals, except that it is customary to make use of *one* angle of incidence only, not the whole range; the reason for this course being that the computations are somewhat simpler if it is adopted. Our next purpose is to derive a formula from which the computations may be carried out in general.

To this end we introduce the notation $E_{\parallel}$ for the component of displacement in the plane of incidence, and $E_{\perp}$ for the component normal to it. Then the ratio of these two is related to the Babinet reading ϕ and the analyzer reading θ by means of the equation

$$\frac{E_{\perp}}{E_{\parallel}} = e^{i\phi} \tan \theta = \rho. \quad (54)$$

This quantity ρ is a complex number, the value of which is known as soon as the experiment has been performed.

Next, we note that the ratio of the normal component of E in the reflected wave to that in the incident wave is given directly by the equations of Table 1. Similarly the ratio of the normal components of H is given directly by those of Table 2; but since both incident and reflected beams are in the same medium, this ratio is the same as that of the components of E which lie in the plane of incidence. From this, and the fact that the two components of E in the incident beam are equal, it follows at once that

$$\rho = \frac{E_{\perp}}{E_{\parallel}} = \frac{\kappa_1 - \kappa}{\kappa_1 + \kappa} \frac{k_1 + k}{k_1 - k}.$$

By remembering that $\kappa_1 = iq_1 c_1 / g_1 = ic_1 h_1 / q_1$ and $k_1 = iq_1 c_1 / h_1$, and by assuming that the permeability of the film is unity, we readily throw this into the form

$$\rho = \frac{qc_1 - cq_1}{qc_1 + cq_1} \frac{q_1 c_1 + qc}{q_1 c_1 - qc} = \frac{qq_1(c_1^2 - c^2) + cc_1(q_1^2 - q^2)}{qq_1(c_1^2 - c^2) - cc_1(q_1^2 - q^2)}.$$

But from (52) we know that

$$c_1^2 - c^2 = r^2(q_1^2 - q^2)/q_1^2,$$

whence we get

$$\rho = \frac{\frac{q}{q_1} r^2 + cc_1}{\frac{q}{q_1} r^2 - cc_1}.$$

Substituting in this equation the value of c_1 as given by (52) and solving for q_1/q we obtain at once the relationship

$$\frac{q_1}{q} = \frac{r}{c(r+1)} [(\rho+1)^2 - 4r^2\rho]^{1/2};$$

or, when we remember that $q_1/q = N + iK_0$, (see §6), and that r and c are the sine and cosine of the angle of incidence,

$$N + iK_0 = \frac{\tan I}{1 + \sin I} [(\rho+1)^2 - 4\rho \sin^2 I]^{1/2}. \quad (55)$$

No matter what angle of incidence we may wish to use, it is only necessary to substitute the measured values in (54), to obtain ρ , and then substitute this ρ in (55). The result will be a complex quantity of which the real part is the index of refraction and the imaginary part is the extinction coefficient, the latter being defined in the manner described in §6. To obtain the coefficient as defined by Drude and Wood we need only make use of the relationship

$$K = K_0/N.$$

§20. *The Determination of the Optical Constants of a Metal by Direct Measurement:* In §15 we called attention to the fact that, since hybrid light does not travel at the speed of ordinary light, there is some question as to whether the angle through which a beam is deflected by a metal prism is governed by the law that applies to dielectric prisms. To resolve these doubts, we may sketch the following solution of the problem of refraction through a prism.

Suppose light is incident at an angle I to the upper surface of a prism. The direction cosines of the velocity vector of the incident and refracted beams are then, by (52)

$$\begin{aligned} a &= \sin I, & a_1 &= \frac{q}{q_1} a, \\ b &= 0, & b_1 &= 0, \\ c &= -\cos I, & c_1 &= -\left(1 - \frac{q^2}{q_1^2} a^2\right)^{1/2}. \end{aligned}$$

We denote the angle of refraction by R (See Fig. 20). If the angle of

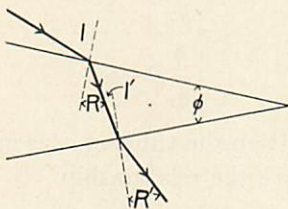


FIG. 20.

the prism is ϕ , the refracted beam meets its lower surface at an angle of incidence $I' = R - \phi$, the direction cosines of which are

$$\begin{aligned} a' &= \sin (R - \phi) = a_1 \cos \phi + c_1 \sin \phi \\ b' &= 0 \\ c' &= -\cos (R - \phi) = -a_1 \sin \phi + c_1 \cos \phi; \end{aligned}$$

and of course the direction cosines of the emergent beam are given at once by (52). They are

$$\begin{aligned} a_1' &= \sin R' = \frac{q_1}{q} a' = a \cos \phi - \left(\frac{q_1^2}{q^2} - a^2 \right)^{1/2} \sin \phi. \\ c_1' &= -\cos R' = -(1 - a_1'^2)^{1/2}. \end{aligned} \tag{56}$$

Now the usual practice in determining the index of refraction is so to orient the prism that $I = -R'$: that is, that the angles of incidence and emergence are equal, but on opposite sides of the normal. This means, however, that $a_1' = -a$, whence (56) may be solved for the relation

$$q_1/q = a/\sin \frac{1}{2}\phi.$$

If we notice that the total deflection of the ray is

$$\delta = (I - R) + (I' - R')$$

and replace I' by $R - \phi$ and R' by $-I$, we obtain the relation

$$2I = \delta + \phi.$$

Finally, since we are dealing with dielectrics, both q and q_1 are real, whence q_1/q is the index of refraction, N . Thus we have

$$N = \sin \frac{1}{2}(\delta + \phi) / \sin \frac{1}{2}\phi. \quad (57)$$

This is the usual formula.

Now, turning to the case of a metal prism, we easily find that a_1' is still equal to $a \cos \phi - (q_1^2/q^2 - a^2)^{1/2} \sin \phi$ as before, and that c_1' is still $-(1 - a_1'^2)^{1/2}$; but these quantities are no longer real, and therefore it is not true that they are the sine and cosine of R' . Instead, this angle is now determined by their *real parts* only. It is possible to write a formula for it, but it is so very complicated that it probably serves no other useful purpose¹² than that of the present investigation, and is scarcely worth reproduction. By finding this angle, however, and setting it equal to $-I$ as before, we can with patience arrive at the result

$$N = \frac{\sin \frac{1}{2}(\delta + \phi)}{\sin \frac{1}{2}\phi} + (K_0^2 - 1) \sin \frac{1}{2}(\delta + \phi) \sin \frac{1}{2}\phi + \frac{N}{N^2 + K_0^2} \sin^2(\delta + \phi) + \dots \quad (58)$$

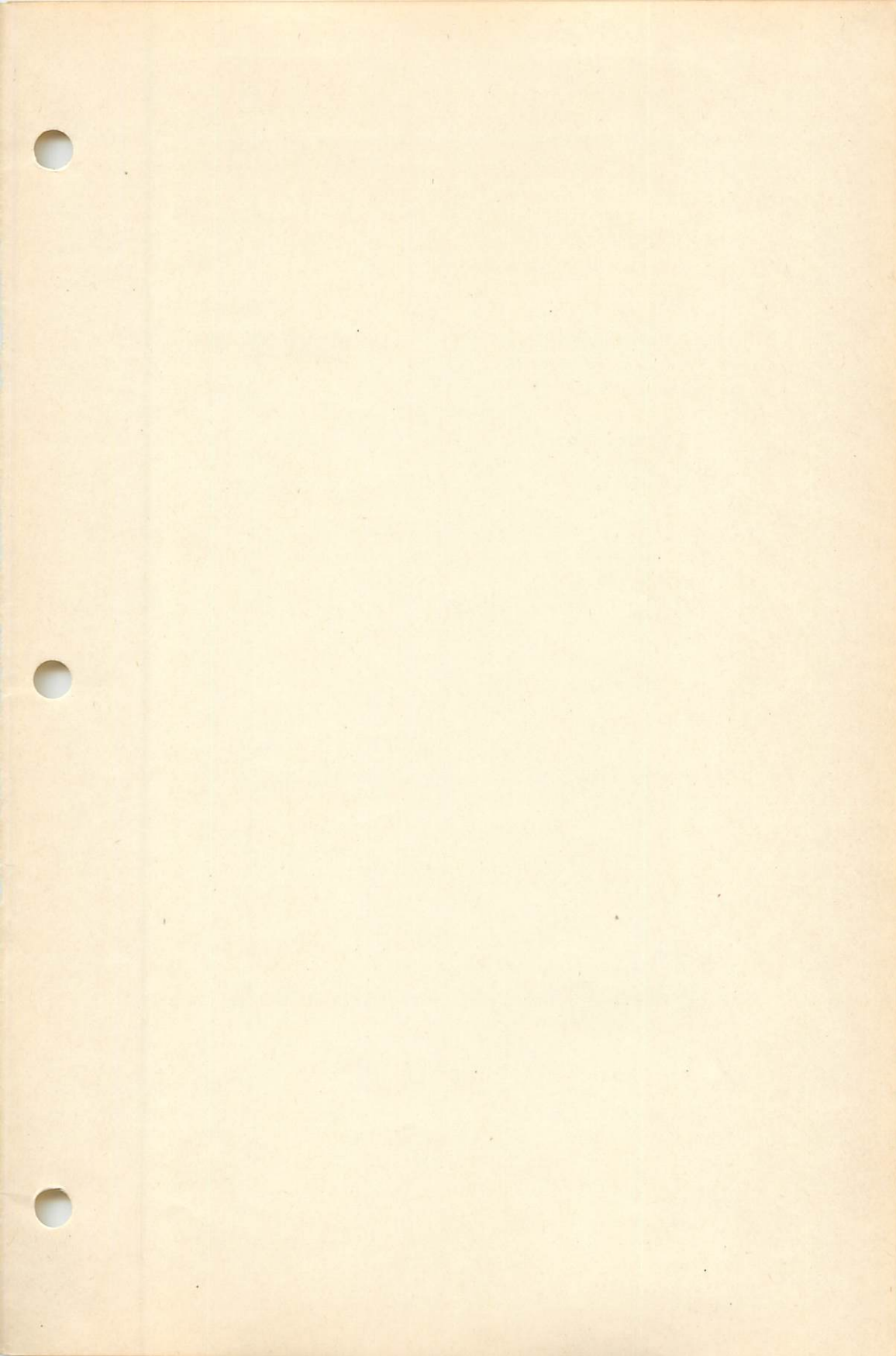
the missing terms being of higher order than the second in the sines.

It follows, then, that the formula *does* depend upon K_0 , and an index of refraction computed in accordance with (57) would be in error if the prism were of metal, as we suspected in §15. But this is true only in a hypercritical sense; for in practice only very thin prisms, for which both δ and ϕ are small, can be used, and for such prisms the terms which involve K_0 are extremely small. In the case of a platinum prism with a 1° angle, for example, the error introduced by using (57) instead of (58) would be only a tenth of a percent. If it were possible to use a high angle prism, however, the error would be considerable: for a 20° prism it would be about 40 percent.

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463 WEST ST., NEW YORK,
JULY 26, 1927.

¹² If we wanted to find the angle of emergence in any particular case we could best do it by computing a_1' and c_1' , and using their real parts only in the formula

$$\tan R' = -a_1'/c_1'.$$



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PLANE WAVES OF LIGHT--III

BY

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A DISCUSSION OF
THE ABSORBING POWER OF BULK METALS
AND OF METALLIC FILMS

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PLANE WAVES OF LIGHT III

ABSORPTION BY METALS

BY THORNTON C. FRY

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§21. *Introduction:* The study which follows has two different aspects. In one sense it is a continuation of two earlier papers¹ bearing the same general title; and as we shall make many cross-references it has seemed wise to run the reference numbers of the present paper consecutively with the former ones. In another sense it is a detail of a more general investigation, the purpose of which is to study the relationship between the optical and photoelectric properties of metals. We begin by indicating its connection with this latter study.

On the experimental side, the general investigation makes much use of thin films of alkali metal deposited either on glass or on a metallic base, frequently platinum. The study of such films, rather than bulk materials, recommends itself for three reasons: first, because they show remarkable variations in photo-sensitivity as the plane of polarization and angle of incidence of the light is varied; second, because the photo-electrons released from bulk material by ordinary light must originate in a very thin layer near the surface, since otherwise they would soon lose their energy by collision; and third, because by working with bases having widely different optical properties, we can distinguish the effects which are due to the photo-sensitive film itself from those due to the nature of the base. This we cannot do if bulk materials are used, though the base (the underlying bulk material) is always present as a possible disturbing factor.

To interpret such experiments adequately, however, we need complete knowledge of the energy distribution within the photo-sensitive layer, and of its relation to the optical properties of the base. This is a purely optical problem, capable of mathematical solution when the optical constants of the materials are known. It is the mathematical study of this optical problem, and in particular of the amount of energy absorbed by the film, with which the present paper deals.

§22. *Physical Background:* The following broad statement of physical background may serve to motivate our mathematical presentation:

¹ Plane Waves of Light, I, J.O.S.A. 15, 137-161; 1927; and II, J.O.S.A. 16, 1-25; 1928

When a beam of light shines upon a metallic plate, the electric intensity inside the plate is proportional to the intensity in the incident beam. The factor of proportionality, however, is not an *absolute* constant. This is obvious since it differs for different metals, and depends on the medium (air or glass, for example) through which the light reaches it. It is not even an *intrinsic* constant of the metal, for if the intensity of the incident beam is kept fixed the electric intensity inside the metal will be different for different planes of polarization, or for different angles of incidence, or for different wave lengths. Moreover, even if the intensity, color and plane of polarization of the incident light are all kept fixed, and only the angle of incidence is allowed to change, the curves which relate the intensity inside the metal to the angle of incidence will be of different shapes at different depths below the surface, due to the fact that the light is extinguished more rapidly the more oblique its angle of incidence. This last point is illustrated by Fig. 21, in which the electric intensity is plotted against angle of incidence at various depths below the surface of a plate of rubidium.

The rate at which energy is absorbed by a metal is, according to the Maxwell theory, directly proportional to the electric intensity *at the place where the absorption takes place*; that is, inside the metal. The constant of proportionality is, in this case, an intrinsic property of the metal. That is, it does not vary with the plane of polarization or angle of incidence of the incident light, nor does it vary from place to place within the metal. It may vary from wave length to wave length, but only if the optical constants also vary.

It must be emphasized that this "*intrinsic absorbing power*," as we shall hereafter call it, is based upon the electric intensity *inside* the metal. It is not what is commonly called the "absorbing power" of the metal. The latter is the ratio of the rate of absorption to the intensity of the *incident* beam, and contains not only the intrinsic factor, but also the other factor of proportionality to which we referred in the first paragraph. It therefore varies with almost every conceivable change in the incident beam. As we shall see in §24, this common "absorbing power" is not primarily governed by the ability of a substance to absorb energy. In fact, it might more appropriately be called a "refracting power," particularly since it depends upon the material through which the light arrives,² as well as that in which absorption takes place. However, in order not to depart too widely from established usage, we shall call it the "bulk absorbing power" of the material, expanding this

² The term "absorbing power" usually implies that the incident beam is in free space.

phrase occasionally to such forms as the "bulk absorbing power of platinum from air," or " . . . from glass," when it is necessary to call particular attention to the medium of incidence.

It does not seem unreasonable to assume that the relationship between the rate at which a metal absorbs light energy and the rate at which it gives off photoelectrons is also an intrinsic property of the metal: that is, that it does not depend at all upon plane of polarization or angle of incidence and that its relation to wave length is not a very complicated

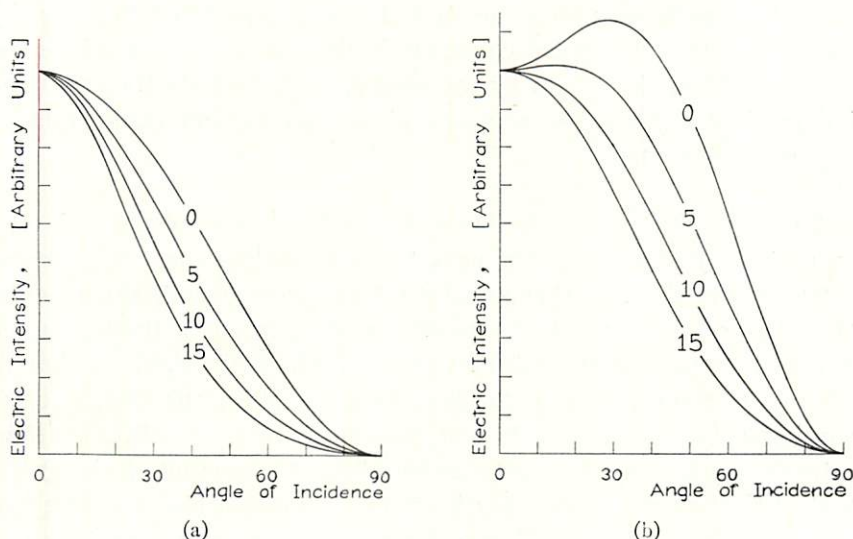


FIG. 21. ELECTRIC INTENSITY VS. ANGLE OF INCIDENCE

The electric vector in (a) is normal to, and in (b) parallel to, the plane of incidence. The optical constants used are those of rubidium at $\lambda_0 = 4893 \text{ \AA}$: $N = 0.134$, $K_0 = 0.868$. The numbers on the curves denote depths (in $\text{cm} \times 10^{-6}$) below the surface of the plate.

one. This hypothesis has neither been proved nor disproved. It merely belongs to the class of things which, if experimentally demonstrated, would be characterized as "what was to have been expected." If it is true, the highly complicated way in which photo-emission varies with color, plane of polarization and angle of incidence must be attributed primarily to the fact that the absorbing power of the metal varies with all these factors, not to any peculiarities of the mechanism of photo-emission. It is even possible that part of the variation of emissivity with wave length may be capable of a similar explanation.

On the other hand, even if the hypothesis is false, the ratio of photo-emission to incident intensity is still the product of three factors: one

having to do with the refractive peculiarities of the metal, a second with its intrinsic absorbing power, and only the third with the mechanism of photo-emission per se. Before we can effectively study the third, therefore, it seems highly desirable to determine the first two independently; by doing this, we shall be able to correct our observations for purely optical influences, and thus arrive at results which can properly be said to represent the *intrinsic photo-emissivity of the metal*.

Such a program is being carried out in the Bell Laboratories by Dr. H. E. Ives and his associates. Some of the experimental results have already been published,³ but the study has so far been handicapped by the lack of reliable optical constants in the spectral region where the work should be done, and by the need for an adequate theory of the purely optical effects. As was said above, the present paper aims to supply the latter need.

For simplicity, we have so far spoken as if we were to deal with only two media. But as the experimental set-up involves three media more often than two, our mathematical study must also include three media: the slab or film in which we are principally interested, the base upon which it rests, and the medium through which the light reaches it. Naturally, the presence of the base does not affect the "intrinsic absorbing power" of the film. It may, however, profoundly affect the light intensity within the film, and therefore cause the amount of energy absorbed to vary with wave length, plane of polarization, and angle of incidence in a way quite different from that observed with only two media. This is particularly true if the layer is so thin that the results of multiple internal reflection are important. In such cases, we shall have to distinguish between the "bulk absorbing power" of an opaque layer and the "lamellar absorbing power" of a thin layer.

We shall find (§24) that when the film is neither nearly transparent nor nearly opaque, the mathematical formulae for this lamellar absorbing power are so involved that their physical interpretation is rather vague. But when the film is almost completely transparent, they separate naturally into factors which are quite easy to interpret; and of course they degenerate into those appropriate to two media when the film becomes opaque.

§23. *Intrinsic Absorbing Power*: The instantaneous rate at which energy is absorbed per unit time per unit volume is given by the formula

³ *The Vectorial Photoelectric Effect in Thin Films of Alkali Metals*, H. E. Ives, Phys. Rev. 38, 1209-1218; 1931; *The Photoelectric Effect From Thin Films of Alkali Metal on Silver*, H. E. Ives and H. B. Briggs, Phys. Rev. 38, 1477-1489; 1931.

$$Q = \sigma E^2 / 4\pi.$$

Here E means the *instantaneous value of the physical electric force*, and is, of course, to be identified with the real component of the complex expressions with which the present paper deals. The question which we raise, therefore, is this:

How much heat energy is absorbed by unit volume of a given substance, in unit time, from a plane wave of unit intensity?

To answer this, we must notice that in the most general case, where E is a rotating vector, it has two components at right angles to one another with amplitudes $E_0 u$ and $E_0 \mathbf{u}$, respectively. (See §7). These components are 90° out of phase with one another: hence they can be written as $E_0 u \cos pt$ and $E_0 \mathbf{u} \sin pt$, respectively. As these components add vectorially, we have

$$Q = \frac{\sigma}{4\pi} E_0^2 (u^2 \cos^2 pt + \mathbf{u}^2 \sin^2 pt).$$

This is the instantaneous rate of absorption. The average rate (time average) is obviously

$$\overline{Q} = \frac{p}{2\pi} \int_0^{2\pi/p} Q dt = \frac{\sigma}{8\pi} E_0^2 (u^2 + \mathbf{u}^2),$$

since the average value of either $\cos^2 pt$ or $\sin^2 pt$ is $1/2$. But, upon recalling the meanings which were attached to the symbols u , $\mathbf{u}$ and E_0 in §7, we readily see that $u^2 + \mathbf{u}^2 = |l^2| + |m^2| + |n^2|$; hence that

$$\overline{Q} = \frac{\sigma}{8\pi} (|E_x^2| + |E_y^2| + |E_z^2|).$$

Finally we may note that, by (8), (10) and footnote 7 (§6),⁴

$$\sigma = 2NK_0 p / \mu$$

whence we have

$$\overline{Q} = \frac{NK_0 p}{4\pi\mu} (|E_x^2| + |E_y^2| + |E_z^2|). \quad (59)$$

Now, the mean square value of the electric force (the "electric intensity," or simply the "intensity") is

$$\overline{E}^2 = \frac{1}{2} (|E_x^2| + |E_y^2| + |E_z^2|);$$

and since we have defined the term "intrinsic absorbing power" to be the ratio $\overline{Q} / \overline{E}^2$, we have

⁴ In this and many equations which follow, we replace the velocity of light, which was denoted by c in §11, by its equivalent $p\lambda_0/2\pi$, in order to avoid confusion with the direction cosine c that appears constantly throughout this part of our work.

$$A_I = \sigma/4\pi = NK_0\rho/2\pi\mu. \quad (60)$$

It should be said in passing that the result (59) may not be so obvious as it seems. Its truth for plane or elliptically polarized light is self-apparent; but not for hybrid light in which the phase relations among the components E_x , E_y and E_z are rather peculiar.

We must also note that the results contained in this section are absolutely general. We are at liberty to apply them to any medium, provided we use the optical constants appropriate to that medium.

§24. *Case of Two Media: Bulk Absorbing Power:* The study of reflection and refraction in the case of two media has already been carried out in §§13–15, the final results being presented in Tables 1 and 2. To apply these results to our present purposes we need only recall that our upper medium is a dielectric. The lower one may be either conductor or dielectric, but it must be “semi-infinite.” Mathematically this means that it must have no other boundary than the plane $z=0$ which separates it from the upper medium. Physically, it implies only that the possibility of light getting back to this plane by internal reflection can be ignored: hence it is satisfied by any opaque body, and also by transparent bodies whose shapes are such as to assure that the refracted beam eventually emerges from other surfaces than the one through which it entered.

We define the “bulk absorbing power” of any such medium by the formula

$$A_B = 1 - |E_z/E|^2; \quad (61)$$

that is, one minus its “reflecting power.” From Table 1 we readily find this to be

$$A_{B\perp} = 1 - \left| \frac{k - k_1}{k + k_1} \right|^2 = \left| \frac{2k}{k + k_1} \right|^2 \operatorname{Re} \frac{k_1}{k}, \quad (62 \perp)$$

when the electric vector is normal to the plane of incidence; and from Table 2

$$A_{B\parallel} = 1 - \left| \frac{\kappa - \kappa_1}{\kappa + \kappa_1} \right|^2 = \left| \frac{2\kappa}{\kappa + \kappa_1} \right|^2 \operatorname{Re} \frac{\kappa_1}{\kappa}, \quad (62 \parallel)$$

when the electric vector is in the plane of incidence.⁵

⁵ The third member of each of these equations is obtained from the second by means of the fundamental identity

$$|a|^2 - |b|^2 = |a + b|^2 \operatorname{Re} \frac{a - b}{a + b}, \quad (63)$$

which is satisfied by any complex numbers a and b . $\operatorname{Re} x$ means the real part of x .

There is another identity of a similar kind which we shall have occasion to use later, and which may best be listed here for reference. It is

$$\operatorname{Re} (1/a) = |1/a|^2 \operatorname{Re} a. \quad (64)$$

To get a physical insight into the meaning of (61) and (62), we go back to the equations listed in §2, and to the long footnote in §6. We there find that⁶

$$h = 2\pi i\mu/\Lambda_0, \quad (65)$$

$$q = (2\pi/\Lambda_0)(N + iK_0). \quad (66)$$

Hence it follows from the definitions of k and κ (Tables 1 and 2) that

$$k = (N + iK_0)c/\mu \quad (67)$$

and, since $q/g = h/q$,

$$\kappa = -\mu c/(N + iK_0). \quad (68)$$

We also note that the total electric force $\mathfrak{E}$ just above the bounding plane $z=0$ is the vector sum of the forces in the incident and reflected beams. In the $\perp$ case the x and z components are zero, and the y components are [by equations (2)] $Eme^{iq(ax+by+cz)-ipt}$ and $E_2M_2e^{iq(a_1x+b_1y+c_1z)-ipt}$, which, upon putting $z=0$ and taking the other quantities from Table 1, add up to

$$\mathfrak{E}_y = (E + E_2)e^{iqrx-ipt};$$

hence

$$\frac{\mathfrak{E}_y}{E} = \frac{2k}{k + k_1}. \quad (69)$$

In the $\parallel$ case there are two components of the electric vector. Treating them just as the y -component was treated above, we find that

$$\frac{\mathfrak{E}_x}{E} = -\frac{2\kappa_1}{\kappa + \kappa_1}\sqrt{1-r^2}, \quad (70)$$

$$\frac{\mathfrak{E}_z}{E} = -\frac{2\kappa}{\kappa + \kappa_1}r. \quad (71)$$

Finally, we may note that the tangential components of the electric vector E_1 just below the bounding surface (that is, at $z=0$) are identical with (69) and (70), as they must be by virtue of Boundary Condition II, §14. The normal component of E_1 , however, is not (71) but

$$\frac{E_{1z}}{E} = -\frac{2\kappa}{\kappa + \kappa_1} \frac{g}{g_1} r.$$

Hence we have

$$E_{1z}/\mathfrak{E}_z = g/g_1. \quad (72)$$

This coefficient g/g_1 defines the discontinuity between the normal

⁶ It need hardly be mentioned that these equations are quite general and therefore apply to any medium provided suitable subscripts are attached. That is, in the lower medium we have $q_1 = 2\pi(N_1 + iK_{10})/\Lambda_0$, etc., the "sub 1" being affixed to the N and K_0 which vary from material to material, but not to Λ_0 , the "ether wave length" of the light, which does not.

components of electric force on the two sides of the boundary.⁷ Though we have derived it from a particular example in which there are only two media and the bounding surface is a plane, it is nevertheless a perfectly general relation, satisfied at all surfaces where two different media meet. Hence we have the rule that:

At the boundary of any two media reflection and refraction of light take place in such a way that the tangential components of the electric vector E , and the normal component of gE , are continuous.

We now return to (62 $\perp$), and note that, by virtue of (69), it reduces to

$$A_{B\perp} = \left| \frac{\mathfrak{E}_\nu}{E} \right|^2 Re \frac{k_1}{k}; \quad (73 \perp)$$

while (62 $\parallel$) can be shown to be equivalent to⁸

⁷ When the media are both dielectrics, g/g_1 reduces to ϵ/ϵ_1 , and (72) becomes the well-known rule that the normal component of displacement is continuous.

If the lower medium is a perfect conductor, g_1 is infinite, and hence $E_{1\perp}=0$, as it should be.

In the more general case, in which both conduction and displacement currents exist, neither of these rules is valid. Instead, the general relation is found from the second of equations (1) to be that $\sigma E_\nu + \epsilon \partial E_\nu / \partial t$ must be continuous; for the tangential components of H , and therefore the z -component of curl H , are known to be continuous. As we are dealing with plane waves in which the time factor is $e^{-i p t}$, this rule requires continuity of $(\sigma - \epsilon i p) E_\nu$; or what amounts to the same thing, of $g E_\nu$.

Finally, we may remark that if the two media are of like permeability, so that $\mu = \mu_1$, g/g_1 is identical with q^2/q_1^2 . This explains the appearance of the factor q^2/q_1^2 in the paper referred to in Footnote 3.

⁸ To establish this, we first note that, by definition,

$$\kappa/\kappa_1 = (c^2 k_1 / c_1^2 k). \quad (74)$$

Also, it follows from Table 2 that

$$c^2/c_1^2 = c^2 + (r^2 q^2 c^2 / q_1^2 c_1^2).$$

We therefore have

$$Re \frac{\kappa}{\kappa_1} = c^2 Re \frac{k_1}{k} + r^2 Re \left(\frac{q^2 c^2}{q_1^2 c_1^2} \frac{k_1}{k} + i \epsilon \right), \quad (75)$$

the term $i \epsilon$ having no effect on the real part of the expression as long as ϵ is a real member. But we may write the last term in the form

$$r^2 Re \left[\frac{k_1}{k} \left(\frac{q^2 c^2}{q_1^2 c_1^2} + i \epsilon \frac{q c}{q_1 c_1} \frac{\mu_1}{\mu} \right) \right], \quad (76)$$

in which form it is easy to give ϵ such a value that the term in parenthesis will be real. To do this, call $q c / q_1 c_1 = u + i v$. Then the parenthesis factors into

$$(u + i v) \left(u + i v + i \epsilon \frac{\mu_1}{\mu} \right)$$

which will obviously be real and equal to $u^2 + v^2 = |q c / q_1 c_1|^2$ provided we set ϵ equal to $-2 v \mu / \mu_1$, as we may do since this value is real. Thus (76) becomes

$$r^2 |q c / q_1 c_1|^2 Re(k_1/k);$$

and hence by (64) and (75),

$$Re \frac{\kappa_1}{\kappa} = c^2 \left| \frac{\kappa_1}{\kappa} \right|^2 Re \frac{k_1}{k} + r^2 \left| \frac{g}{g_1} \right|^2 Re \frac{k_1}{k}.$$

$$A_{B\parallel} = \left(\left| \frac{\mathfrak{E}_x}{E} \right|^2 + \left| \frac{g}{g_1} \right|^2 \left| \frac{\mathfrak{E}_x}{E} \right|^2 \right) \operatorname{Re} \frac{k_1}{k}. \quad (73||)$$

The interpretation of these equations scarcely needs comment: in each case the first factor is the ratio of the intensity just below the surface to the intensity of the incident beam, while the second factor plays a role somewhat similar to the "intrinsic absorbing power" in equation (59). But it is not the intrinsic absorbing power, nor indeed any *intrinsic* property of the medium, as is evident from the fact that it varies with the angle of incidence.

In fact, the "bulk absorbing powers" (62), or (73), are not truly named at all. They are in reality phenomena of *refraction*, not of *absorption*. Whatever succeeds in passing the bounding surface is absorbed sooner or later, if the lower medium is opaque, or passes out of the system if it is transparent. The amount "absorbed" is therefore determined solely by the reflecting power of the metal; not by its subsequent disposal of the refracted portion.

However, as it is this quantity which is spoken of as "absorbing power" in classical optics, it would probably only create confusion to replace it by a more appropriate term.

§25. *Case of Three Media: Lamellar Absorbing Power:* We now turn our attention to the case of three media, separated by the parallel planes $z=0$ and $z=-\zeta$ as shown in Fig. 22, which is, except for minor modifications, a reproduction of Fig. 18.

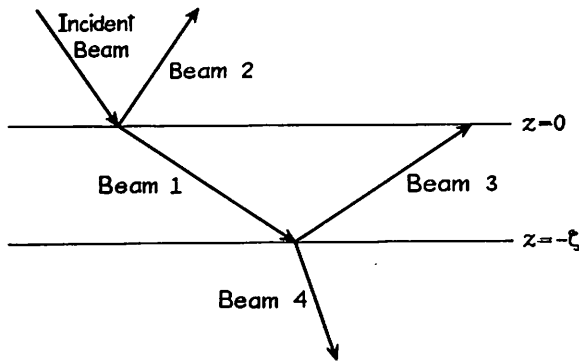


FIG. 22.

In our physical application to the problem of photoelectric emission, the middle substance will ordinarily be a thin film of alkali metal, the bottom one the base upon which it is deposited, and the upper one air or vacuum. It will, therefore, save a great deal of circumlocution if we refer to them simply as "film," "base" and "air." However, the results which we shall obtain will be valid no matter what materials are used for

film and base, provided only that the upper medium is a transparent dielectric. In other words, unless exceptions are explicitly noted, the only restrictions are that all three materials shall be isotropic and that the extinction coefficient of the upper one shall be zero.

Moreover, so far as the present section is concerned, the film need not be "thin." It can have any thickness from zero (which is a case of two media) to infinity (which is again a case of two media).

A beam of plane-polarized light is incident upon the upper surface of the film. Part of it is reflected as Beam 2, part is transmitted into the base as Beam 4 and the rest is absorbed by the film, in which there are a refracted Beam 1 and an internally reflected Beam 3.

Symbols which refer to these various beams will be supplied with corresponding subscripts. Thus, if they have no subscripts, they refer to the Incident Beam, while if they have subscripts 1, 2, 3 or 4 they refer

TABLE 3. *Reflection and refraction from a film. E normal to the plane of incidence.*

$l=0$	$a=r$	$l'=-c$
$m=1$	$b=0$	$m'=0$
$n=0$	$c=-(1-r^2)^{1/2}$	$n'=a$
$l_1=0$	$a_1=q_1r/q_1$	$l_1'=-c_1$
$m_1=1$	$b_1=0$	$m_1'=0$
$n_1=0$	$c_1=-(1-q_1^2r^2/q_1^2)^{1/2}$	$n_1'=a_1$
$l_2=0$	$a_2=r$	$l_2'=-c_2$
$m_2=1$	$b_2=0$	$m_2'=0$
$n_2=0$	$c_2=+(1-r^2)^{1/2}$	$n_2'=a_2$
$l_3=0$	$a_3=q_3r/q_1$	$l_3'=-c$
$m_3=1$	$b_3=0$	$m_3'=0$
$n_3=0$	$c_3=+(1-q_3^2r^2/q_1^2)^{1/2}$	$n_3'=a_3$
$l_4=0$	$a_4=q_4r/q_4$	$l_4'=-c_4$
$m_4=1$	$b_4=0$	$m_4'=0$
$n_4=0$	$c_4=-(1-q_4^2r^2/q_4^2)^{1/2}$	$n_4'=a_4$

$$\begin{aligned}
 \frac{E_1}{E} &= \frac{2k(k_1 + k_4)X^2}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \\
 \frac{E_2}{E} &= \frac{(k - k_1)(k_1 + k_4)X^2 + (k + k_1)(k_1 - k_4)}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \\
 \frac{E_3}{E} &= \frac{2k(k_1 - k_4)}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}, \\
 \frac{E_4'}{E} &= \frac{4kk_1X}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)}.
 \end{aligned}
 \tag{77 \perp}$$

Notes:

1. The symbol k is used for the combination iqc/h , the q , c , and h all being understood to have the same subscripts as the k .

2. The symbol X is used for $e^{iq_1z_1}\zeta$.

3. It is not necessary to write the H 's. They are defined by (9), which is valid for any medium, provided q , h , E and H all have the subscript appropriate to that medium.

4. The signs affixed to the radicals denote the signs of the real parts.

to Beams 1, 2, 3 or 4.⁹ In addition, since we have no occasion to discuss the values of the various electric and magnetic forces at other places than the bounding surfaces, we shall understand all such symbols to denote values at the upper surface $z=0$ unless a prime is attached, in which case they denote values at $z=-\zeta$.

In order to facilitate reference, we assemble in Tables 3 and 4 the information needed to define the amplitudes, directions of propagation,

TABLE 4. *Reflection and refraction from a film. H normal to the plane of incidence.*

$l=c$	$a=r$	$l'=0$
$m=0$	$b=0$	$m'=1$
$n=-a$	$c=-(1-r^2)^{1/2}$	$n'=0$
$l_1=c_1$	$a_1=q_1r/q_1$	$l_1'=0$
$m_1=0$	$b_1=0$	$m_1'=1$
$n_1=-a_1$	$c_1=-(1-q^2r^2/q_1^2)^{1/2}$	$n_1'=0$
$l_2=c_2$	$a_2=r$	$l_2'=0$
$m_2=0$	$b_2=0$	$m_2'=1$
$n_2=-a_2$	$c_2=+(1-r^2)^{1/2}$	$n_2'=0$
$l_3=c_3$	$a_3=q_3r/q_1$	$l_3'=0$
$m_3=0$	$b_3=0$	$m_3'=1$
$n_3=-a_3$	$c_3=+(1-q^2r^2/q_1^2)^{1/2}$	$n_3'=0$
$l_4=c_4$	$a_4=q_4r/q_4$	$l_4'=0$
$m_4=0$	$b_4=0$	$m_4'=1$
$n_4=-a_4$	$c_4=-(1-q^2r^2/q_4^2)^{1/2}$	$n_4'=0$

$$\begin{aligned}
 \frac{H_1}{H} &= \frac{2\kappa(\kappa_1 + \kappa_4)X^2}{(\kappa + \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa - \kappa_1)(\kappa_1 - \kappa_4)}, \\
 \frac{H_2}{H} &= \frac{(\kappa - \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa + \kappa_1)(\kappa_1 - \kappa_4)}{(\kappa + \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa - \kappa_1)(\kappa_1 - \kappa_4)}, \\
 \frac{H_3}{H} &= \frac{2\kappa(\kappa_1 - \kappa_4)}{(\kappa + \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa - \kappa_1)(\kappa_1 - \kappa_4)}, \\
 \frac{H_4'}{H} &= \frac{4\kappa\kappa_1X}{(\kappa + \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa - \kappa_1)(\kappa_1 - \kappa_4)}.
 \end{aligned} \tag{77}$$

⁹ Certain quantities which measure intrinsic properties of the three substances necessarily have the same values for the Incident Beam and Beam 2, and also for Beams 1 and 3. The index

Notes:

1. The symbol κ is used for the combination iqc/g , the q , c , and g being understood to have the same subscripts as the κ .
2. The symbol X is used for $e^{iq_1c} \zeta$.
3. It is not necessary to write the E 's. They are defined by (9) which is valid for any medium, provided g , h , E and H all have the subscript appropriate to that medium.
4. The signs affixed to the radicals denote the signs of the real parts.

damping, and states of polarization of the various beams. This material is derived from §§14–17, and its interpretation should be clear in the light of the extended comment of §§2–10. The form of presentation is identical with that used in Tables 1 and 2.

With this information before us, we are in a position to compute the z -components of the Poynting vectors (or rather the time-average of these components) for the Beams 1, 2 and 4. As explained in §11, these represent, respectively, the amount of incident energy which falls per second upon a square centimeter of the upper surface, the amount carried away from this square centimeter by the reflected beam, and the amount carried through a square centimeter of the lower surface into the base. If we call these S_z , S_{2z} and S_{4z} , respectively, it is obvious that each square centimeter of illuminated film must absorb $|S_z| - |S_{2z}| - |S_{4z}|$ units of energy per second. The *proportion* absorbed, or the *lamellar absorbing power of the film*, is

$$A_L = 1 - \frac{|S_{2z}| + |S_{4z}|}{|S_z|}.$$

However, since the energy-flow in the incident and transmitted beams is *downward* (that is, in the negative z -direction), while that in the reflected beam is *upward*, the signs of S_z and S_{4z} will be negative, and that of S_{2z} positive. Hence, when algebraic signs are fully accounted for,

$$A_L = 1 + \frac{S_{2z}}{S_z} - \frac{S_{4z}}{S_z}. \quad (78)$$

We now proceed to the evaluation of S_z , S_{2z} and S_{4z} , taking the $\perp$ case first.

As the upper medium is a dielectric, the electric and magnetic vectors are in phase and hence

$$S_z = (p\Lambda_0/16\pi^2) [EH]_z,$$

of refraction N , the extinction coefficient K_0 and the propagation constant q belong to this class. In the case of such quantities the subscript 1 is used consistently for the film, even when we are talking about the internally reflected Beam 3, and no subscript at all for air, even when dealing with Beam 2.

where the brackets denote the vector product. But, in the $\perp$ case, $[EH]_z = [EH_z] = |EH|c$, and therefore

$$S_z = (p\Lambda_0/16\pi^2) |EH|c.$$

We may remark in passing that, by the use of (8), (9) and (10), it is possible to throw this into the alternative forms

$$S_z = (p\Lambda_0/16\pi^2) |E^2|k = - (p\Lambda_0/16\pi^2) |H^2|k, \quad (79)$$

which we shall have occasion to use later on. It also follows that, if the beam is of unit intensity, (that is, if $|E|^2 = 2$) the amount of energy which falls upon one square centimeter of the plane $z=0$ is by (67)¹⁰

$$- p\Lambda_0 k/8\pi^2 = - p\Lambda_0 Nc/8\pi^2 \mu.$$

By an exactly similar argument we find the z -component of the Poynting vector in the reflected beam to be

$$S_{2z} = - (p\Lambda_0/16\pi^2) |E_2^2|k = (p\Lambda_0/16\pi^2) |H_2^2|k. \quad (80)$$

Dividing this equation by (79), and replacing $|E_2/E|^2$ by its value as given in Table 3, we have¹¹

$$\frac{S_{2z}}{S_z} = - \left| \frac{(k - k_1)(k_1 + k_4)X^2 + (k + k_1)(k_1 - k_4)}{(k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4)} \right|^2. \quad (81 \perp)$$

This result has been obtained for the $\perp$ case. It is physically obvious, however, that (79) and (80) must be true for the other case also, since the relationship of the intensity of the light to the magnitude of its electric force-vector does not depend upon the plane of polarization; and this conclusion is easily verified by direct computation of S_z and S_{2z} from the data of Table 4. If, then, we again divide (80) by (79) and replace $|H_2/H|^2$ by its value as given in Table 4, we get the lamellar reflecting power of the film for the $\parallel$ case. It is

$$\frac{S_{2z}}{S_z} = - \left| \frac{(\kappa - \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa + \kappa_1)(\kappa_1 - \kappa_4)}{(\kappa + \kappa_1)(\kappa_1 + \kappa_4)X^2 + (\kappa - \kappa_1)(\kappa_1 - \kappa_4)} \right|^2. \quad (81 \parallel)$$

¹⁰ The negative sign is explained as follows: Since the beam is directed downward, c , k , and therefore S_z also, are negative. But by "amount of energy" we evidently mean an essentially positive quantity. Hence we must replace the negative numbers c and k by positive numbers $-c$ and $-k$.

¹¹ This is the lamellar reflecting power of the film in the presence of the base. It reduces to the bulk reflecting power of the base when the thickness of the film is made zero (this corresponds to $X=1$), and to the bulk reflecting power of the film material alone, provided the thickness of the film is made infinite (this corresponds to $X=0$). As we have said, the negative sign merely reflects the fact that the energy flows downward in the incident beam and upward in the reflected one.

We are now ready to turn our attention to the light which passes into the base. Here the situation is a bit more complicated owing to the fact that this beam is "hybrid." Again, however, we may follow the procedure laid down in §11.

We take first the $\perp$ case, and note that, since E_4' lies in the y -direction, we need only use the x -component of the magnetic force in our computation of S_{4x} . However, by Table 3, the x -component of magnetic force is readily found to be $H_4'l_4'$, which, by (9) and Table 3 is easily seen to be given by

$$H_4'l_4'/E_4' = -iq_4c_4/h_4 = -k_4.$$

If the base is a metal, k_4 is a complex quantity.¹² Calling it

$$k_4 = Re^{i\eta},$$

the argument of §11 shows that

$$\begin{aligned} S_{4x} &= (p\Lambda_0/16\pi^2) |E_4'H_4'l_4'| \cos \eta \\ &= (p\Lambda_0/16\pi^2) |E_4'|^2 R \cos \eta \\ &= (p\Lambda_0/16\pi^2) |E_4'|^2 Re k_4. \end{aligned}$$

Dividing this by (79), and taking the value of $|E_4'^2/E^2|$ from Table 3, we get

$$\frac{S_{4x}}{S_z} = \left| \frac{4kk_1X}{(k+k_1)(k_1+k_4)X^2 + (k-k_1)(k_1-k_4)} \right|^2 Re(k_4/k). \quad (82 \perp)$$

This formula applies to the $\perp$ case. For the $\parallel$ case we find without great difficulty that

$$\frac{S_{4x}}{S_z} = \left| \frac{4\kappa\kappa_1X}{(\kappa+\kappa_1)(\kappa_1+\kappa_4)X^2 + (\kappa-\kappa_1)(\kappa_1-\kappa_4)} \right|^2 Re(\kappa_4/\kappa). \quad (82 \parallel)$$

By substituting (81) and (82) in (78) we find the absorbing power of the film in the presence of its backing plate. The formula thus obtained, however, may be considerably simplified by means of certain tedious though not especially difficult algebraic transformations. As they are purely formal, and involve no physical concepts whatever, we shall omit them and state our result at once in the final form which appears to be most convenient for purposes of calculation:

¹² This is also true even when the base is a dielectric, if total reflection takes place at the lower boundary of the film. Our argument, though nominally carried out for a metallic base, is good for dielectric bases also, whether totally reflecting or not, since real values of k_4 merely imply that $\eta=0$.

$$A_{L\perp} = 2 \left[\left| (k_1 + k_4)X + (k_1 - k_4) \right|^2 \left| k(X + 1) \right|^2 \operatorname{Re} \frac{k_1(X - 1)}{k(X + 1)} \right. \\ \left. + \left| (k_1 + k_4)X - (k_1 - k_4) \right|^2 \left| k(X - 1) \right|^2 \operatorname{Re} \frac{k_1(X + 1)}{k(X - 1)} \right] \\ \div \left| (k + k_1)(k_1 + k_4)X^2 + (k - k_1)(k_1 - k_4) \right|^2. \quad (83 \perp)$$

The formula for $A_{L\parallel}$ is identical, except that all k 's are replaced by κ 's. We shall refer to it as (83 $\parallel$).

This equation, then, gives the lamellar absorbing power of the film for any angle of incidence, for any thickness of film and (with the k 's replaced by κ 's) for either plane of polarization of the incident light. It is valid whether the film and base are conductors or dielectrics. It is even possible for the film and base to be of identical material, in which case (83) gives the proportion of the incident radiation which is absorbed by a layer of this material adjacent to the bounding surface.

§26. *Approximate Formulae: Lamellar Absorbing Power of Very Thin Films:* We have already indicated that the purposes of this study have to do primarily with very thin films, so thin in fact that they are virtually transparent. For such films we can derive approximate formulae which are both simpler than the ones given in §25, and also easier to interpret physically.

To this end, it is convenient to introduce the notation.

$$\epsilon = iq_1c_1\zeta = k_1h_1\zeta, \quad (84)$$

and assume that this ϵ is a small quantity.¹³ Then we have approximately

$$X \doteq 1, \quad X - 1 \doteq \epsilon, \quad X + 1 \doteq 2; \quad (85)$$

and therefore (83 $\perp$) becomes

$$A_{L\perp} \doteq \left| \frac{2k}{k + k_4} \right|^2 \left[\operatorname{Re} \frac{k_1\epsilon}{k} + \left| \frac{k_4\epsilon}{k_1} \right|^2 \operatorname{Re} \frac{k_1}{k\epsilon} \right]. \quad (86 \perp)$$

¹³ The physical significance of this assumption is easily seen. As explained in §§4 and 6, the real part of $iq_1c_1\zeta$ measures the amount by which light is attenuated in its travel from the upper to the lower surface of the film. The imaginary part measures the phase change that takes place in the same vertical distance (which is less than the phase change which takes place in the same distance when measured in the direction of propagation). To say that ϵ is small, therefore, means that the film is thin enough to be nearly transparent, and also that it is thin when compared with the wave length of the light in the film (not its ether wave length). In order to meet the latter restriction ζ ought not exceed some such value as $\Lambda/20$.

In the experimental study with which the present paper is mainly concerned, the film is usually a monomolecular layer of an alkali metal. For such films the conditions are probably met for ether wave lengths greater than 100A.

By an identical argument we also get from (83 ||)

$$A_{L||} \doteq \left| \frac{2\kappa}{\kappa + \kappa_4} \right|^2 \left[Re \frac{\kappa_1 \epsilon}{\kappa} + \left| \frac{\kappa_4 \epsilon}{\kappa_1} \right|^2 Re \frac{\kappa_1}{\kappa \epsilon} \right]. \quad (86 ||)$$

The formal analogy between the $\perp$ and $||$ cases ends at this point, and we shall therefore find it most convenient to drop the $||$ case for a time and concentrate upon the study of (86 $\perp$).

To begin with, we must recall that the upper medium is a dielectric. Hence by (67)

$$k = Nc/\mu$$

is real. Therefore, upon using (66), (67) and (84), we find without difficulty that

$$Re(k_1/k\epsilon) = 0,$$

$$Re(k_1\epsilon/k) = -4\pi\mu N_1 K_{10} \zeta / c\mu_1 \Lambda_0 N.$$

Finally, noting that $-c$ is the cosine of the angle of incidence, (86 $\perp$) takes the form

$$A_{L\perp} \doteq \frac{\left(\frac{N_1 K_{10} p}{2\pi\mu_1} \right) \left(\left| \frac{2k}{k + k_4} \right|^2 \right) \zeta}{\left(\frac{N\Lambda_0 p \cos I}{8\pi^2\mu} \right)}. \quad (87 \perp)$$

For purposes of physical interpretation (87 $\perp$) has been divided into four factors. We see from (60) that the first in the numerator is the intrinsic absorbing power of the material of which the film is composed. That is, it is the rate at which heat energy is produced in unit volume of film material by a wave of unit electric intensity, *this intensity being measured inside the film*.

The interpretation of the second factor is clear from (69). It is the ratio of the total intensity *just above the surface of the film* to the intensity of the incident beam, computed as if the film were absent and the reflection took place directly from the base. It need hardly be noted that by Boundary Condition II, §14, the intensity *just below the surface* is the same as that just above.

The factor in the denominator is the rate at which energy is carried to unit area of film by an incident beam of unit intensity.

The product of these three factors, then, gives the ratio of the energy absorbed per unit volume of film to the amount falling upon unit area of film.

Finally the fourth factor, ζ , is the volume of film material per square centimeter of exposed area.

Turning now to the $\parallel$ case and applying the transformation (64) to the last term, we readily throw (86 $\parallel$) into the form

$$A_{L\parallel} \doteq \left| \frac{2\kappa}{\kappa + \kappa_4} \right|^2 Re \frac{\kappa_1 \epsilon}{\kappa} + \left| \frac{2\kappa_4}{\kappa + \kappa_4} \right|^2 Re \frac{\kappa \epsilon}{\kappa_1}. \quad (88)$$

But by using (84); (65), (66), (67) and (68); and then finally (8), we readily find that

$$Re \frac{\kappa_1 \epsilon}{\kappa} = r^2 \left| \frac{g}{g_1} \right|^2 \left(\frac{N_1 K_{10} p}{2\pi \mu_1} \right) \left(\frac{8\pi^2 \mu}{N \Lambda_0 p \cos I} \right) \zeta,$$

and

$$Re \frac{\kappa \epsilon}{\kappa_1} = (1 - r^2) \left(\frac{N_1 K_{10} p}{2\pi \mu_1} \right) \left(\frac{8\pi^2 \mu}{N \Lambda_0 p \cos I} \right) \zeta.$$

Inserting these in (88), and arranging the terms somewhat after the fashion of those in (87 $\perp$), we get finally:

$$A_{L\parallel} \doteq \frac{\left(\frac{N_1 K_{10} p}{2\pi \mu_1} \right) \left(\left| \frac{2\kappa_4}{\kappa + \kappa_4} \right|^2 (1 - r^2) + \left| \frac{g}{g_1} \right|^2 \left| \frac{2\kappa}{\kappa + \kappa_4} \right|^2 r^2 \right) \zeta}{\left(\frac{N \Lambda_0 p \cos I}{8\pi^2 \mu} \right)}. \quad (87 \parallel)$$

We have again four factors, of which three are identical with three in (87 $\perp$). The meaning of the other is easily seen by reference to (70), (71) and (72). It contains the components of electric intensity *just above the film*, computed as if the film were absent, with the normal component corrected for the discontinuity that must take place on passing the boundary.

The net result of the argument then is this:

In the case of thin¹⁴ films, the lamellar absorbing power can be found to a high degree of accuracy by

(a) *ignoring the presence of the film and finding the components of total electric intensity that would exist just above the surface if reflection took place directly between air and BASE [by using (69), (70) and (71)];*

(b) *correcting the normal component for the discontinuity which must take place in crossing the surface between air and FILM [by using (72)];*

(c) *finding the rate at which energy is absorbed by the film material from a wave of this amplitude [by use of the intrinsic absorbing power (60) and the thickness ζ];*

¹⁴ That is, thin enough to justify the use of the approximations (85). (See Footnote 13.)

(d) and finally dividing this by the rate at which energy is conveyed to the film by the incident beam.

In other words, both (87 $\perp$) and (87 $\parallel$) can be written in a single form:

$$A_L \doteq \frac{\left(\frac{N_1 K_{10} p}{2\pi\mu_1}\right) \left(\left| \frac{\mathcal{E}_x}{E} \right|^2 + \left| \frac{\mathcal{E}_y}{E} \right|^2 + \left| \frac{g}{g_1} \right|^2 \left| \frac{\mathcal{E}_z}{E} \right|^2 \right) \zeta}{\left(\frac{N\Lambda_0 p \cos I}{8\pi^2\mu} \right)}, \quad (89)$$

which applies to *any* plane of polarization; it being understood that the total electric intensities $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$ are those which would exist *if reflection took place directly from the base*.

It may be noted in passing that the only factors of (89) that depend upon the character of the film material are its intrinsic absorbing power and the discontinuity ratio g/g_1 , neither of which is in any way dependent upon the angle of incidence of the light, or upon its state of polarization.

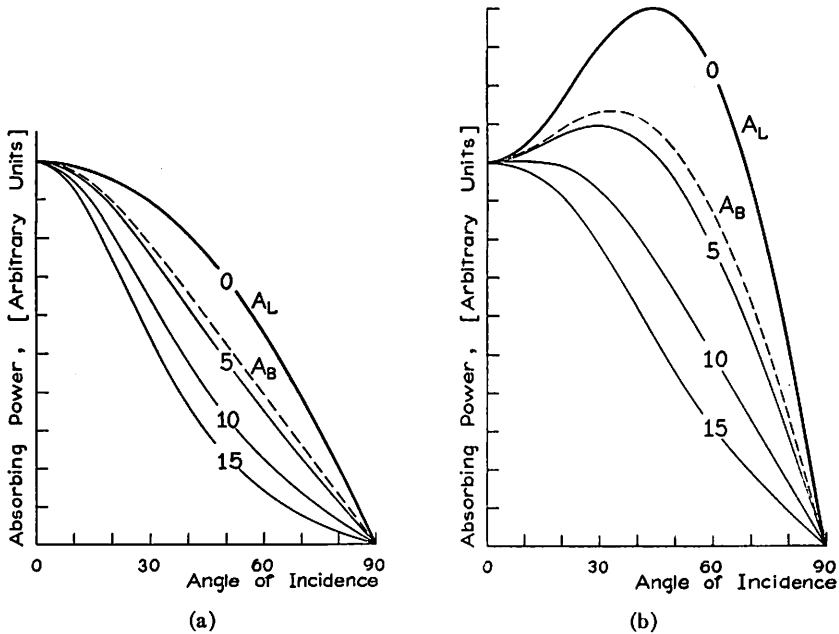


FIG. 23. ABSORBING POWER VS. ANGLE OF INCIDENCE

The electric vector in (a) is normal to, and in (b) parallel to, the plane of incidence.

The optical constants used are those of rubidium at $\Lambda_0 = 4893 \text{ \AA}$: $N = 0.134$, $K_0 = 0.868$.

The dotted curve is bulk absorbing power; the heavy curve lamellar absorbing power of rubidium on rubidium; and the lighter curves the lamellar absorbing powers of thin layers at depths below the surface (in $\text{cm} \times 10^{-6}$) indicated by the numbers on the curves.

§27. *Some Numerical Illustrations:* As a first illustration of the use to which the formulae of §§23–26 may be put, we may compute the bulk absorbing power of rubidium, using the optical constants $N=0.134$, $K_0=0.868$ corresponding to wave length $\Lambda_0=4893\text{\AA}$. For this purpose, of course, we make use of equations (62 $\perp$) and (62 $\parallel$) of §24.

The result is shown by the dotted curves of Fig. 23 for the two planes of polarization.

As a second example, we may consider the lamellar absorbing power of rubidium on rubidium: that is, the absorbing power of a very thin layer near the surface of a rubidium plate. It is given by equation (89) of §26, provided we assign the constants of rubidium to both the film and the base.

The only factors in this expression which vary with the angle of incidence are $|\mathcal{E}_x/E|^2 + |\mathcal{E}_y/E|^2 + |g/g_1|^2 |\mathcal{E}_z/E|^2$ and $\cos I$. The other factors are unimportant so long as we are willing to plot our results in arbitrary units. But the factor $|\mathcal{E}_x/E|^2 + |\mathcal{E}_y/E|^2 + |g/g_1|^2 |\mathcal{E}_z/E|^2$ is the electric intensity *just below the surface*, which we have already plotted as the curves marked 0 in Fig. 21. To get the lamellar absorbing power of rubidium on rubidium, therefore, we need only multiply these upper curves of Fig. 21 by the secant of the angle of incidence, thus getting the heavy curves of Fig. 23. The deviation from the bulk absorbing power is striking.

We may go a step further in this connection. We know that the rate at which energy is absorbed by a thin layer *at any depth within the rubidium plate* is proportional to the electric intensity at that depth: that is, to the various curves of Fig. 21. If, then, we divide each of these curves by $\cos I$, (which enters (87) in computing the rate at which energy falls on the surface) we will obtain a set of curves showing what proportion of the incident energy is absorbed at various depths. Plotted in arbitrary units, so as more clearly to show their different behavior as the angle of incidence changes, they are given by the light curves of Fig. 23. Calling these, for the moment, “lamellar absorbing powers at various depths,” we may state the rather obvious fact that the curve for bulk absorbing power versus angle of incidence is a sort of average among the lamellar curves at various depths.¹⁵

In saying that this is obvious, however, I cannot refrain from adding that the dissimilarity of the lamellar curves themselves is a consequence of the fact that light is damped out more rapidly when incident obliquely

¹⁵ If we had kept the scale factors in mind, instead of plotting everything in arbitrary units as we have done in Fig. 23, the bulk curve would be just the integral of the lamellar curves *with respect to depth*.

than when incident normally; and I do not believe *this* fact is either obvious or well known.

Digressing for a moment to the subject of photoemission, we may remark that *if a correlation is to be expected at all between the photo emissivity of bulk material and its optical absorbing power, we would expect it to be found in connection with the dark lamellar curve rather than the dotted bulk curve*; since the electrons emitted at appreciable depths below the surface will probably never emerge.

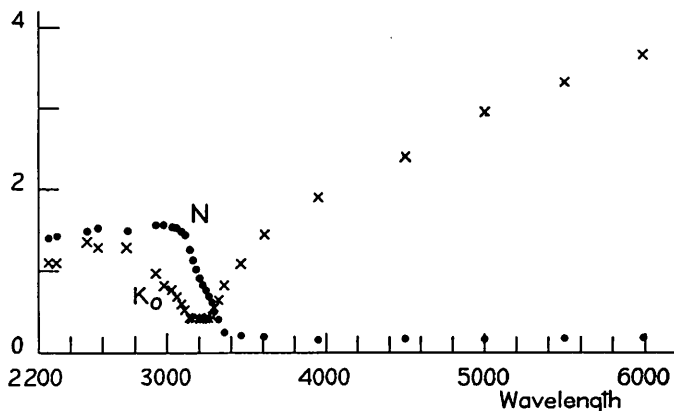


FIG. 24. Optical constants of silver (Minor's values).

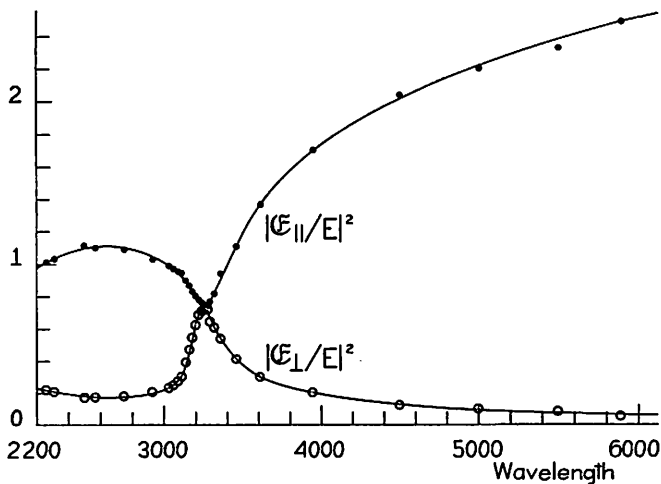


FIG. 25. Total electric intensity vs. wave length. $\mathcal{E}$ is the total electric force just above the reflecting surface due to the superposition of incident and reflected beams. Minor's optical constants for silver were used.

As a third example, we may consider the variation of absorbing power with wave length. Silver is one of the few metals for which the optical

constants have been reliably determined over a broad spectral region, and fortunately also presents an interesting case from our point of view. In Fig. 24 we give the values of N and K_0 as determined by Minor between $\Lambda_0 = 2263\text{\AA}$ and $\Lambda_0 = 5893\text{\AA}$. Using these values in (69), (70) and (71), and confining our attention to a 60° angle of incidence, we readily obtain the curves of Fig. 25 for the electric intensities just above the silver surface (that is, $|\mathcal{E}_\perp/E|^2 = |\mathcal{E}_y/E|^2$ for the $\perp$ case, and $|\mathcal{E}_\parallel/E|^2 = |\mathcal{E}_x/E|^2 + |\mathcal{E}_z/E|^2$ for the $\parallel$ case).

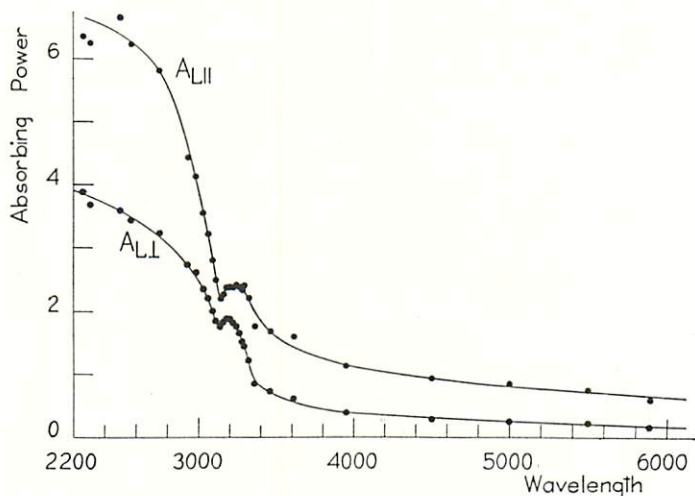


FIG. 26. Lamellar absorbing power of silver on silver. Computed from Minor's constants. For a film of thickness ξ the vertical scale readings are to be multiplied by $10^5 \xi$.

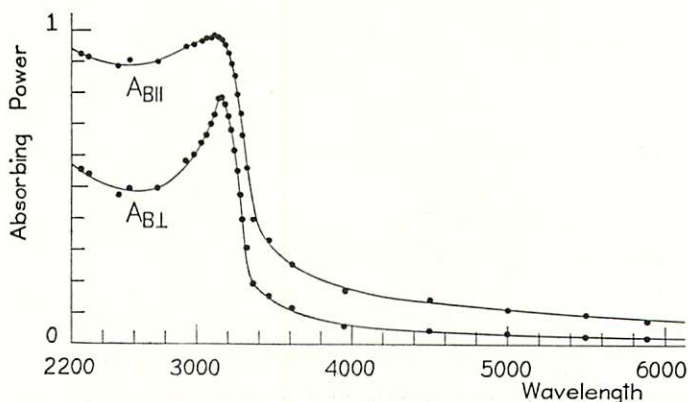


FIG. 27. Bulk absorbing power of silver. Computed from Minor's constants.

Next we obtain the lamellar absorbing powers as given by (87 $\perp$) and (87 $\parallel$). We can no longer neglect the denominator and the first factor

in the numerator, as we did in computing Fig. 23, since N_1 , K_{10} and Λ_0 are all variable. The curves marked $A_{L\perp}$ and $A_{L\parallel}$ on Fig. 26, therefore, give quantitatively the proportion of incident light which is absorbed by a thin surface layer of the silver mirror from which it is reflected, so long as the thickness of the layer does not exceed the limits defined in Footnote 13. The scale readings are to be multiplied by $10^5\%$.

Finally, we substitute the optical constants of Fig. 24 in (73 $\perp$) and (73 $\parallel$) and obtain the bulk absorbing power of silver over the same spec-

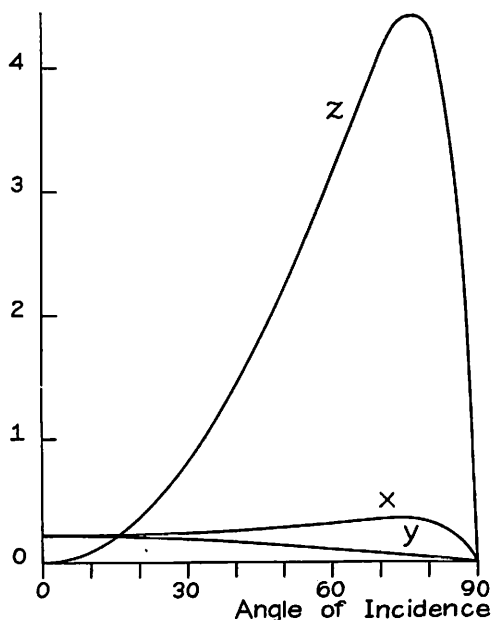


FIG. 28.

The x, y and z components of $|E/E|^2 \sec I$ vs. angle of incidence.

The y curve is for $\perp$ light; the x and z curves for $\parallel$ light.

The optical constants $N=1.928$, $K_0=3.175$ correspond to platinum at $\Lambda_0=4359 \text{ \AA}$.

tral region. These are shown in Fig. 27. The great dissimilarity between lamellar and bulk absorption is again apparent.

As a fourth illustration we may take an example in which the film and base are of different substances. For this case we again discuss the variation of absorbing power with angle of incidence, a platinum base with the constants $N_4=1.928$, $K_{40}=3.175$, and a wave length $\Lambda_0=4359 \text{ \AA}$. Ignoring the presence of the film, and using (69), (70) and (71), we compute the quantities $|E_y/E|^2 \sec I$, $|E_x/E|^2 \sec I$ and $|E_z/E|^2 \sec I$, thus getting the results shown in Fig. 28. Except for the factor $\sec I$, these represent electric intensities *just above* the surface, the first one referring

to the case of $\perp$ polarization and the other two to the $\parallel$ case. The factor $\sec I$ has been introduced at this point purely for convenience: it

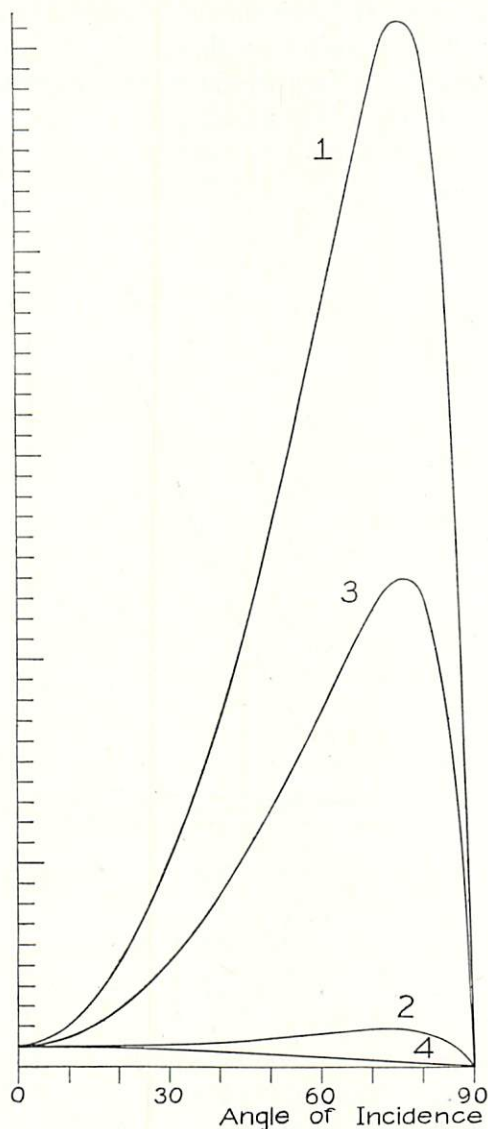


FIG. 29.

- (1) Lamellar absorbing power of potassium on platinum ($\parallel$ case).
- (2) Lamellar absorbing power of platinum on platinum ($\parallel$ case).
- (3) $|\mathcal{E}_{\parallel}/E|^2 \sec I$.
- (4) Lamellar absorbing power of potassium on platinum, or of platinum on platinum ($\perp$ case);
or $|\mathcal{E}_{\perp}/E|^2 \sec I$.

The vertical scale is in arbitrary units.

occurs in every subsequent formula to which we refer, and we save computation by introducing it as early as possible.

In accordance with (87 $\parallel$) we may get the lamellar absorbing power $A_{L\parallel}$ of any thin film deposited on this platinum base by multiplying $|\mathcal{E}_z/E|^2 \sec I$ by the $|g/g_1|^2$ appropriate to the *film* material, and adding it to $|\mathcal{E}_x/E|^2 \sec I$. There is in addition, of course, a scale factor composed of various N 's, K 's and Λ_0 's; but this does not vary with the angle

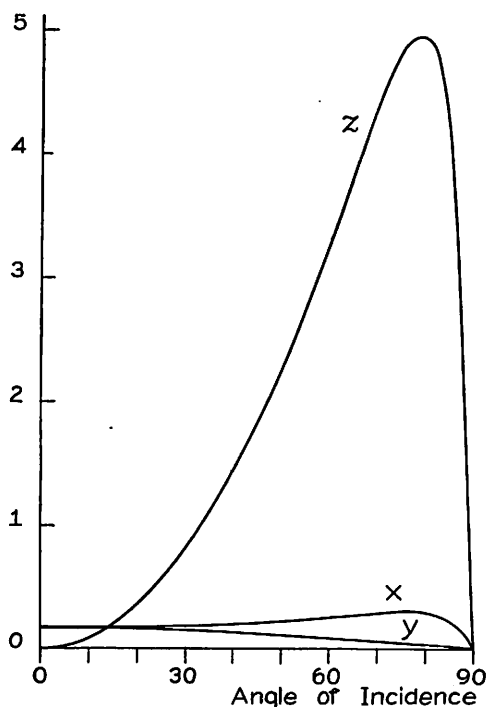


FIG. 30.

The x , y and z components of $|\mathcal{E}/E|^2 \sec I$ vs. angle of incidence.

The y curve is for $\perp$ light; the x and z curves for $\parallel$ light.

The optical constants $N=2.438$, $K_0=3.451$ correspond to platinum at $\Lambda_0=5461 \text{ \AA}$.

of incidence and is of no importance so long as we are interested only in the functional relationship of $A_{L\parallel}$ to I . Similarly, by (87 $\perp$), the lamellar absorbing power $A_{L\perp}$ is (except for a scale factor) independent of the material of the film.

Using these ideas, we plot on Fig. 29 the lamellar absorbing power $A_{L\perp}$ which applies to any substance, and the $A_{L\parallel}$ for films of potassium¹⁶ and of platinum. At this wave length the ratio of selective to normal lamel-

¹⁶ The constants (adopted from Duncan's data) are $N_1=0.0690$; $K_{10}=0.815$.

lar absorption is nearly twice as high for a thin film of potassium on platinum, as for the surface layer of platinum itself.

For comparison we also plot the curves

$$|\mathcal{E}_\perp/E|^2 \sec I = |\mathcal{E}_y/E|^2 \sec I, \quad |\mathcal{E}_\parallel/E|^2 \sec I = (|\mathcal{E}_x/E|^2 + |\mathcal{E}_z/E|^2) \sec I$$

which represent the intensity just above the surface multiplied by $\sec I$.

As an illustration of the great changes that may be produced in

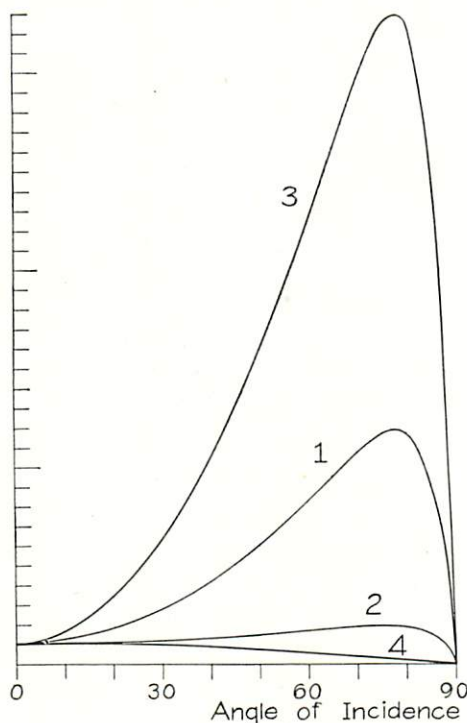


FIG. 31.

- (1) Lamellar absorbing power of potassium on platinum ($\parallel$ case).
- (2) Lamellar absorbing power of platinum on platinum ($\parallel$ case).
- (3) $|\mathcal{E}_\parallel/E|^2 \sec I$.
- (4) Lamellar absorbing power of potassium on platinum, or of platinum on platinum ($\perp$ case) or $|\mathcal{E}_\perp/E|^2 \sec I$.

The vertical scale is in arbitrary units.

lamellar absorption by relatively small changes in the optical constants of the film we include also Figs. 30 and 31, which are identical with Figs. 28 and 29 except that they refer to a wave length $\Lambda_0 = 5461\text{\AA}$. Though the platinum constants $N = 2.438$, $K_0 = 3.451$, are now somewhat different than before, they produce little change in Fig. 30, in the

$A_{L\perp}$ curve, (Fig. 31) or in the $A_{L\parallel}$ curve for a platinum film (Fig. 31). But the $A_{L\parallel}$ curve for a potassium film rises only half as high as the $\mathcal{E}_{\parallel}$ curve in Fig. 31, whereas in Fig. 29 it was twice as high. The constants for potassium, $N_1=0.0691$; $K_{10}=1.325$, were again taken from Duncan's data.

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